

A relative higher index theorem, diffeomorphisms and positive scalar curvature

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Abstract

We prove a general relative higher index theorem for complete manifolds with positive scalar curvature towards infinity. We apply this theorem to study Riemannian metrics of positive scalar curvature on manifolds. For every two metrics of positive scalar curvature on a closed manifold and a Galois cover of the manifold, we define a secondary higher index class. Non-vanishing of this higher index class is an obstruction for the two metrics to be in the same connected component of the space of metrics of positive scalar curvature. In the special case where one metric is induced from the other by a diffeomorphism of the manifold, we obtain a formula for computing this higher index class. In particular, it follows that the higher index class lies in the image of the Baum-Connes assembly map.

1 Introduction

In this paper, we use methods from noncommutative geometry to study problems of positive scalar curvature on manifolds. From the work of Fomenko and Mischenko [21], Kasparov [15], and Connes and Moscovici [9], methods from noncommutative geometry have found many impressive applications towards geometry and topology, in particular, to those related to the Novikov conjecture and the positive scalar curvature problem. The fact that the positive scalar curvature problem is closely related to the Novikov conjecture (or the Baum-Connes conjecture) was already made apparent by Rosenberg in [22]. Block and Weinberger [6], and the second author [24][25] successfully applied noncommutative geometric methods to determine the existence (nonexistence) of positive scalar curvature on certain classes of manifolds. By applying the work of Lott on higher eta invariants (which

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is noncommutative geometric) [20], Leichtnam and Piazza studied the connectedness of the space of all Riemannian metric of positive scalar curvature on certain classes of manifolds [18].

One of main tools used in all the studies mentioned above is index theory in the context of noncommutative geometry, often referred as *higher index theory*. The method of applying (classical) index theory to study the positive scalar curvature problem on manifolds goes back to Lichnerowicz. By applying the Atiyah-Singer index theorem [1], he showed that a compact spin manifold does not support positive scalar curvature if its \hat{A} -genus is nonzero [19]. With a refined version of the Atiyah-Singer index theorem [2], Hitchin showed that half of the exotic spheres in dimension 1 and 2 (mod 8) cannot carry metrics of positive scalar curvature [13]. This line of development was pursued further by Gromov and Lawson. In [11], they developed a relative index theorem and obtained nonexistence of positive scalar curvature for a large class of (not necessarily compact) manifolds. In [8], Bunke proved a relative higher index theorem and applied it to study problems of positive scalar curvature on manifolds.

In this paper, we prove a general relative higher index theorem (for both real and complex cases). We apply this theorem to study Riemannian metrics of positive scalar curvature on manifolds. For every two metrics of positive scalar curvature on a closed manifold and a Galois cover of the manifold, there is a naturally defined secondary higher index class. Non-vanishing of this higher index class is an obstruction for the two metrics to be in the same connected component of the space of metrics of positive scalar curvature. In the special case where one metric is induced from the other by a diffeomorphism of the manifold, we obtain a formula for computing this higher index class. In particular, it follows that the higher index class lies in the image of the Baum-Connes assembly map.

It is essential to allow real C^* -algebras and their (real) K -theory groups when studying problems of positive scalar curvature on manifolds, cf. [11][22]. In fact, the (real) K -theory groups of real C^* -algebras provide more refined invariants for obstructions of existence of positive scalar curvature. We point out that the proofs in our paper are written in such a way that they apply to both the real and the complex cases. In order to keep the notation simple, we shall only prove the results for the complex case and indicate how to modify the arguments, if needed, for the real case. From now on, unless otherwise specified, all bundles and algebras are defined over \mathbb{C} .

Here is a synopsis of the main results of the paper. Let X_0 and X_1 be

two even dimensional¹ spin manifolds with complete Riemannian metrics of positive scalar curvature (uniformly bounded below) away from compact sets. Assume that we have compact subspaces $K_i \subset X_i$ such that there is an (orientation preserving) isometry $\Psi : \Omega_0 \rightarrow \Omega_1$, where $\Omega_i \subset X_i - K_i$ is a union of (not necessarily all) connected components of $X_i - K_i$ (see Figure 1 in Section 4). We emphasize that the Riemannian metric on Ω_i may have *nonpositive* scalar curvature on some compact subset. Let \mathcal{S}_i be the corresponding spinor bundle over X_i . We assume that Ψ lifts to a bundle isometry $\tilde{\Psi} : \mathcal{S}_0|_{\Omega_0} \rightarrow \mathcal{S}_1|_{\Omega_1}$. Let $(X_i)_\Gamma$ be a Γ -cover² of X_i , where Γ is a discrete group. We assume that Ψ lifts to an isometry on the covers. Let D_i be the associated Dirac operator on $(X_i)_\Gamma$. Then we have $D_1 = \tilde{\Psi} \circ D_0 \circ \tilde{\Psi}^{-1}$ on $(\Omega_1)_\Gamma$.

Let N be a compact hypersurface in $\Omega \cong \Omega_i$ such that N cuts X_i into two components. We separate off the component that is inside Ω_i and denote the remaining part of X_i by Y_i (see Figure 1 in Section 4). We obtain X_2 by gluing Y_0 and Y_1 along N . Moreover, we glue the spinor bundles over Y_0 and Y_1 to get a spinor bundle over X_2 . All these cutting-pastings lift to the covers, and produce a Γ -cover $(X_2)_\Gamma$ of X_2 . Let D_2 be the associated Dirac operator on $(X_2)_\Gamma$. For each D_i , we have its higher index class $\text{Ind}(D_i) \in K_0(C_r^*(\Gamma))$ (resp. $K_0(C_r^*(\Gamma; \mathbb{R}))$ in the real case). We have the following relative higher index theorem (Theorem 4.2).

Theorem A.

$$\text{Ind}(D_2) = \text{Ind}(D_0) - \text{Ind}(D_1).$$

The usefulness of the above theorem lies in the fact that the index class on the left hand side is computable in many cases (for example, when X_2 is compact), while the index classes on the right hand side are difficult to compute. In the proof of this theorem, we carry out a construction of invertible doubles (Theorem 5.1). Our construction takes place on manifolds with $C_r^*(\Gamma)$ -bundles³ (resp. $C_r^*(\Gamma; \mathbb{R})$ -bundles) and generalizes the invertible double construction for manifolds with classical vector bundles (i.e. \mathbb{C} -vector bundles or \mathbb{R} -vector bundles), cf. [7, Chapter 9]. We point out that if the scalar curvature on Ω is positive everywhere, then our theorem above follows from Bunke's relative higher index theorem [8, Theorem 1.2]. In our

¹In the real case, we assume $\dim X_0 = \dim X_1 \equiv 0 \pmod{8}$.

²All covering spaces considered in this paper are Galois covering spaces, i.e. regular covering spaces.

³For a real or complex C^* -algebra \mathcal{A} , by a \mathcal{A} -bundle over a manifold M , we mean a locally trivial Banach vector bundle over M whose fibers have the structure of finitely generated projective \mathcal{A} -modules.

theorem, we allow the scalar curvature on Ω to be nonpositive on a compact subset. In particular, in the case when X_0 and X_1 are both compact, our theorem applies to any Riemannian metrics (possibly with scalar curvature nowhere positive) on X_0 and X_1 .

As an application of our relative higher index theorem, we consider a compact odd dimensional⁴ spin manifold M (with a fixed spin structure) which supports positive scalar curvature. Let M_Γ an Γ -cover of M , where Γ is a discrete group. Let Ψ be an orientation preserving diffeomorphism $\Psi : M \rightarrow M$ which in addition preserves the spin structure of M . Choose $g_0 \in \mathcal{R}^+(M)$. Let $g_1 = (\Psi^{-1})^*g_0$ and g_t a smooth path of Riemannian metrics on M with

$$g_t = \begin{cases} g_0 & \text{for } t \leq 0, \\ g_1 & \text{for } t \geq 1, \\ \text{any smooth homotopy from } g_0 \text{ to } g_1 & \text{for } 0 \leq t \leq 1. \end{cases}$$

Then $X = M \times \mathbb{R}$ endowed with the metric $h = g_t + (dt)^2$ becomes a complete Riemannian manifold with positive scalar curvature away from a compact set. Let $X_\Gamma = M_\Gamma \times \mathbb{R}$ and D the corresponding Dirac operator on X_Γ . Then we have the higher index class $\text{Ind}(D) \in K_0(C_r^*(\Gamma))$ (resp. $\text{Ind}(D) \in K_0(C_r^*(\Gamma; \mathbb{R}))$ in the real case).

Assume that Ψ lifts to a diffeomorphism $\tilde{\Psi} : M_\Gamma \rightarrow M_\Gamma$. This is always the case when $\Gamma = \pi_1(M)$ the fundamental group of M . Notice that Ψ induces an automorphism $\Gamma \rightarrow \Gamma$. Let $\Gamma \rtimes \mathbb{Z}$ be the semi-direct product with the action of \mathbb{Z} on Γ induced by Ψ . Then $X_\Gamma = M_\Gamma \times \mathbb{R}$ becomes a $(\Gamma \rtimes \mathbb{Z})$ -cover of M_Ψ . Here $M_\Psi = (M \times [0, 1]) / \sim$, where \sim is the equivalence relation $(x, 0) \sim (\Psi(x), 1)$ for $x \in M$. We denote by $D_{\Gamma \rtimes \mathbb{Z}}$ the Dirac operator on X_Γ , which defines a higher index class $\text{Ind}(D_{\Gamma \rtimes \mathbb{Z}}) \in K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$ (resp. $\text{Ind}(D_{\Gamma \rtimes \mathbb{Z}}) \in K_0(C_r^*(\Gamma \rtimes \mathbb{Z}; \mathbb{R}))$ in the real case), cf. [9, Section 5]. Now let $\iota : \Gamma \hookrightarrow \Gamma \rtimes \mathbb{Z}$ be the natural inclusion map, which induces a homomorphism $\iota_* : K_0(C_r^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$. Then we have the following theorem.

Theorem B.

$$\iota_*(\text{Ind}(D)) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$$

in $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$ (resp. $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}; \mathbb{R}))$ for the real case).

If we assume that the strong Novikov conjecture holds for $\Gamma \rtimes \mathbb{Z}$, then the above theorem provides a formula to determine when $\iota_*(\text{Ind}(D))$ is non-vanishing. Note that the above theorem implies that $\iota_*(\text{Ind}(D))$ lies in

⁴In the real case, we assume $\dim M \equiv -1 \pmod{8}$.

the image of the Baum-Connes assembly map $\mu : K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$ (or its real analogue [5]). It remains an open question whether $\text{Ind}(D) \in K_0(C_r^*(\Gamma))$ lies in the image of the Baum-Connes assembly map $\mu : K_0^{\Gamma}(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma))$.

An outline of the paper is as follows. In Section 2, we review some basic facts in K -theory and index theory. In Section 3, we discuss some basic properties of Dirac operators (in Hilbert modules over a C^* -algebra) and construct their higher index classes (with finite propagation property). In Section 4, we prove a general relative higher index theorem. In Section 5, we carry out an invertible double construction. In Section 6, we apply our relative higher index theorem to study positive scalar curvature problem on manifolds under diffeomorphisms.

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2 Preliminaries

In this section, we review some preliminary facts in K -theory and index theory.

2.1 Abstract index theory

Let \mathcal{B} be a unital C^* -algebra and

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I} \rightarrow 0$$

a short exact sequence of C^* -algebras. Then we have the following six-term⁵ long exact sequence in K -theory.

$$\begin{array}{ccccc} K_0(\mathcal{I}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{B}/\mathcal{I}) \\ \text{Ind} \uparrow & & & & \downarrow \text{Ind} \\ K_1(\mathcal{B}/\mathcal{I}) & \longleftarrow & K_1(\mathcal{B}) & \longleftarrow & K_1(\mathcal{I}) \end{array}$$

In particular, each invertible element in $u \in \mathcal{B}/\mathcal{I}$ defines an element $\text{Ind}(u) \in K_0(\mathcal{I})$, called the index of u .

⁵In the real case, the long exact sequence has 24 terms, due to the fact real K -theory has periodicity 8.

We recall an explicit construction of the index map (in the even case). Let $u \in \mathcal{B}$ such that u becomes invertible in \mathcal{B}/\mathcal{I} . Then there exists $v \in \mathcal{B}$ such that both $uv - 1$ and $vu - 1$ are in \mathcal{I} . We define

$$w = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that w is invertible and a direct computation shows that

$$w - \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_2(\mathcal{I}) = M_2(\mathbb{C}) \otimes \mathcal{I}.$$

Consider the idempotent

$$p = w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} = \begin{pmatrix} uv + uv(1 - uv) & (2 + uv)(1 - uv)u \\ v(1 - uv) & (1 - uv)^2 \end{pmatrix}.$$

We have

$$p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{I})$$

and

$$\text{Ind}(u) = [p] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(\mathcal{I}).$$

Now suppose $F = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \in M_2(\mathcal{B})$ is a self-adjoint element such that $F^2 - 1 \in M_2(\mathcal{I})$. In this case, the index of F is defined to be

$$\text{Ind}(F) := \text{Ind}(u).$$

2.2 K-theory and almost idempotents

There are several equivalent ways to define the K -theory groups of a C^* -algebra. In this subsection, we review a slightly non-standard definition, which will be used in the later sections.

In the following discussion, we fix a (sufficiently small) constant $\tau > 0$ once and for all.

Definition 2.1. Let \mathcal{B} be a C^* -algebra and \mathcal{I} a closed ideal of \mathcal{B} .

- (a) We call an element $x \in \mathcal{B}$ is τ -close to \mathcal{I} if there exists an element $y \in \mathcal{I}^+$ such that

$$\|x - y\| < \min\{\tau, \|x\|^{-1}\tau\}.$$

(b) An element $z \in \mathcal{B}$ is called a τ -almost idempotent if

$$\|z^2 - z\| < \tau.$$

Notice that if z is a τ -almost idempotent, then (as long as τ is sufficiently small) we can choose disjoint open sets U and V such that $\text{spec}(z) \subset U \cup V$ with $0 \in U$ and $1 \in V$. Set $h = 0$ on U and $h = 1$ on V and define

$$p = \frac{1}{2\pi i} \int_C \frac{h(\zeta)}{\zeta - z} d\zeta$$

where C is a contour surrounding $\text{spec}(z)$ in $U \cup V$. Then p is an idempotent and therefore defines a K -theory class.

Now let p be an idempotent in $M_\infty(\mathcal{B})$. In general, such an idempotent does not define a K -theory class in $K_0(\mathcal{I})$. However, if p is τ -close to \mathcal{I} (with τ sufficiently small), then p does uniquely define an element in $K_0(\mathcal{I})$. Indeed, choose $q \in M_\infty(\mathcal{I}^+)$ so that $\|p - q\| < \min\{\tau, \|p\|^{-1}\tau\}$. Since $p^2 - p = 0$, we have

$$\|q^2 - q\| \leq \|(p - q)p\| + \|q(p - q)\| + \|p - q\| < 4\tau,$$

i.e. $q \in M_\infty(\mathcal{I}^+)$ is a (4τ) -almost idempotent. By the above discussion, q defines a K -theory class in $K_0(\mathcal{I}^+)$. Let π be the quotient map

$$\pi : \mathcal{I}^+ \rightarrow \mathcal{I}^+/\mathcal{I} = \mathbb{C}.$$

and $\pi_* : K_0(\mathcal{I}^+) \rightarrow K_0(\mathbb{C}) = \mathbb{Z}$ the induced map on K_0 -groups. If $\pi_*([q]) = [n]$, then we have $[q] - [n] \in K_0(\mathcal{I})$. We shall still denote this class by $[p] - [n]$ if no confusion arises.

Remark 2.2. Notice that the class $[p] - [n] \in K_0(\mathcal{I})$ does not depend on the choice of q . Indeed, if we choose another $q' \in M_\infty(\mathcal{I}^+)$ such that $\|p - q'\| < \min\{\tau, \|p\|^{-1}\tau\}$, then we have $\|q - q'\| < 2\min\{\tau, \|p\|^{-1}\tau\}$. It is easy to verify that $[q] = [q'] \in K_0(\mathcal{I}^+)$.

2.3 A difference construction

In this subsection, we review the difference construction in K -theory from [16, Section 6]. Let \mathcal{B} be a C^* -algebra and \mathcal{I} be a two-sided closed ideal in \mathcal{B} . For each pair of idempotents $p, q \in \mathcal{B}$ with $p - q \in \mathcal{I}$, we shall define a difference element $E(p, q) \in K_0(\mathcal{I})$.

First consider the invertible element

$$Z(q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

whose inverse is

$$Z(q)^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix}.$$

Then we define

$$E_0(p, q) = Z(q)^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(q).$$

A direct computation shows that

$$E_0(p, q) = \begin{pmatrix} 1+q(p-q)q & 0 & qp(p-q) & 0 \\ 0 & 0 & 0 & 0 \\ (p-q)pq & 0 & (1-q)(p-q)(1-q) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

It follows immediately that $E_0(p, q) \in M_4(\mathcal{I}^+)$ and $E_0(p, q) = e$ modulo $M_4(\mathcal{I})$, where

$$e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Definition 2.3. Define

$$E(p, q) = [E_0(p, q)] - [e] \in K_0(\mathcal{I}).$$

Remark 2.4. In fact, the same construction works when $(p-q)$ is τ -close to \mathcal{I} . In this case, although $E_0(p, q) - e \notin M_4(\mathcal{I})$, the explicit formula (1) shows that $E_0(p, q)$ is τ -close to \mathcal{I} (with a slight modification of the definition of τ -closeness). According to the discussion in the previous subsection, $E_0(p, q)$ defines a K -theory class in $K_0(\mathcal{I})$, which we shall still denote by $E(p, q)$.

3 Dirac operators and their higher index classes

In this section, we review some basic properties of Dirac operators over Galois covers of complete manifolds and their higher index classes.

Let X be a complete even dimensional spin manifold. Let X_Γ be a Γ -cover of X . We define a flat $C_r^*(\Gamma)$ -bundle⁶ \mathcal{V} on X by

$$\mathcal{V} = X_\Gamma \times_\Gamma C_r^*(\Gamma),$$

where Γ acts on X_Γ and $C_r^*(\Gamma)$ diagonally. Denote by $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the spinor bundle over X and

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the grading operator on \mathcal{S} . Then with the flat connection of \mathcal{V} , we can define the Dirac operator

$$D_{\mathcal{V}} : \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$$

where $\Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$ is the space of smooth sections of $\mathcal{S} \otimes \mathcal{V}$ over X . We will simply write D for $D_{\mathcal{V}}$ if no ambiguity arises. With the \mathbb{Z}_2 -grading on \mathcal{S} , we have

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Here are some standard properties of this Dirac operator D as a unbounded operator on the $C_r^*(\Gamma)$ -Hilbert module $\mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ the space of L^2 sections of $\mathcal{S} \otimes \mathcal{V}$ over X :

- (a) D is an essentially self-adjoint operator;
- (b) $D^2\sigma = 0 \Leftrightarrow D\sigma = 0$, for $\sigma \in \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$;
- (c) if the scalar curvature κ of the manifold X is uniformly bounded, then the maximal domain of D on $\mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ is exactly the Soblev space $\mathcal{H}^1(X, \mathcal{S} \otimes \mathcal{V})$;
- (d) $D^2 = \nabla^*\nabla + \frac{\kappa}{4}$, where $\nabla : \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X, T^*X \otimes \mathcal{S} \otimes \mathcal{V})$ is the connection on the bundle $\mathcal{S} \otimes \mathcal{V}$ and ∇^* is the adjoint of ∇ .

⁶In the real case, we consider the bundle $\mathcal{V} = X_\Gamma \times_\Gamma C_r^*(\Gamma; \mathbb{R})$.

From now on, let us assume that X has (*strictly*) *positive scalar curvature towards infinity*. More precisely, there exist a subset $\Omega \subset X$ such that $X - \Omega$ is compact and the Riemannian metric has positive scalar curvature $> k_0$ on Ω , for some positive constant k_0 .

In this case, there exists a compactly supported function $\rho \geq 0$ on X such that

- (i) $\frac{\kappa}{4} + \rho^2 \geq c_0 > 0$ for some fixed constant c_0 ,
- (ii) $\|[D, \rho]\|$ as small as we want, in particular $< \frac{c_0}{2}$.

Lemma 3.1. $D_{\mathcal{V}} + \varepsilon\rho : \mathcal{H}^1(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ is bounded below.

Proof. Indeed, for each $\sigma \in \mathcal{H}^1(X, \mathcal{S} \otimes \mathcal{V})$, we have

$$\begin{aligned}
\|(D_{\mathcal{V}} + \varepsilon\rho)\sigma\|_0^2 &= \| \langle (D + \varepsilon\rho)\sigma, (D + \varepsilon\rho)\sigma \rangle \| \\
&= \| \langle (D^2 + [D, \rho]\varepsilon + \rho^2)\sigma, \sigma \rangle \| \\
&\geq \| \langle (\nabla^*\nabla + \frac{\kappa}{4} + \rho^2)\sigma, \sigma \rangle \| - \frac{c_0}{2} \|\sigma\|_0^2 \\
&= \| \langle \nabla\sigma, \nabla\sigma \rangle + \langle (\frac{\kappa}{4} + \rho^2)\sigma, \sigma \rangle \| - \frac{c_0}{2} \|\sigma\|_0^2 \\
&\geq c \| \langle \nabla\sigma, \nabla\sigma \rangle + \langle \sigma, \sigma \rangle \| = c \|\sigma\|_{\mathcal{H}^1}^2
\end{aligned}$$

where $\|\cdot\|_0$ denotes the L^2 -norm on $\mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ and $c = \min\{1/2, c_0/2\} > 0$. We have used the fact that $a \geq b \geq 0 \Rightarrow \|a\| \geq \|b\|$ in a C^* -algebra. \square

It follows that $D_{\mathcal{V}} + \varepsilon\rho$ has a bounded inverse

$$(D_{\mathcal{V}} + \varepsilon\rho)^{-1} : \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \mathcal{H}^1(X, \mathcal{S} \otimes \mathcal{V}).$$

3.1 Generalized Fredholm operators in Hilbert modules

Let \mathcal{A} be a C^* -algebra and $\mathcal{H}_{\mathcal{A}}$ a Hilbert module over \mathcal{A} . We denote the space of all adjointable operators in $\mathcal{H}_{\mathcal{A}}$ by $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. Let $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ be the space of all compact adjointable operators in $\mathcal{H}_{\mathcal{A}}$. Note that $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$.

Definition 3.2. (cf. [23, Chapter 17]) An operator $F \in \mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is called a generalized Fredholm operator if $\pi(F) \in \mathcal{B}(\mathcal{H}_{\mathcal{A}})/\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ is invertible.

Proposition 3.3. *Let*

$$F = D_{\mathcal{V}}(D_{\mathcal{V}}^2 + \rho^2 + [D_{\mathcal{V}}, \rho]\varepsilon)^{-1/2} : \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V}).$$

Then F is a generalized Fredholm operator.

Proof. Let $D = D_{\mathcal{V}}$ and

$$T = (D_{\mathcal{V}}^2 + \rho^2 + [D_{\mathcal{V}}, \rho]\varepsilon)^{-1/2} = \sqrt{\frac{1}{(D_{\mathcal{V}} + \varepsilon\rho)^2}}.$$

First, we show that $F : \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ is bounded. Indeed,

$$\begin{aligned} \|F\sigma\|_0^2 &= \|\langle DT\sigma, DT\sigma \rangle\| \\ &= \|\langle (TD^2T\sigma, \sigma) \rangle\| \\ &= \|\langle \sigma - T(\rho^2 + [D, \rho]\varepsilon)T\sigma, \sigma \rangle\| \\ &\leq \|1 - R\| \|\sigma\|_0^2 \end{aligned}$$

where $R = T(\rho^2 + [D, \rho]\varepsilon)T : \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \mathcal{L}^2(X, \mathcal{S} \otimes \mathcal{V})$ is compact, in particular bounded.

To show that F is a generalized Fredholm operator, it suffices to show that $F^2 - 1$ is compact, i.e. $F^2 - 1 \in \mathcal{K}(\mathcal{H}_{C^*(\Gamma)})$. Since

$$\frac{1}{\sqrt{x}} = \int_0^\infty \frac{1}{x^2 + \lambda^2} d\lambda,$$

we have

$$T = \sqrt{\frac{1}{(D + \varepsilon\rho)^2}} = \int_0^\infty ((D + \varepsilon\rho)^2 + \lambda^2)^{-1} d\lambda.$$

Now notice that

$$\begin{aligned} &((D + \varepsilon\rho)^2 + \lambda^2)^{-1} D - D((D + \varepsilon\rho)^2 + \lambda^2)^{-1} \\ &= ((D + \varepsilon\rho)^2 + \lambda^2)^{-1} [D, \rho^2 + [D, \rho]\varepsilon] ((D + \varepsilon\rho)^2 + \lambda^2)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} F^2 &= DTD T \\ &= D \left(\int_0^\infty ((D + \varepsilon\rho)^2 + \lambda^2)^{-1} d\lambda \right) D T \\ &= \left(D^2 \int_0^\infty ((D + \varepsilon\rho)^2 + \lambda^2)^{-1} d\lambda + D \int_0^\infty K(\lambda) d\lambda \right) T \\ &= D^2 T^2 + D K T \\ &= 1 - (\rho^2 + [D, \rho]\varepsilon) T^2 + D K T^2 \end{aligned}$$

where

$$K(\lambda) = ((D + \varepsilon\rho)^2 + \lambda^2)^{-1} [D, \rho^2 + [D, \rho]\varepsilon] ((D + \varepsilon\rho)^2 + \lambda^2)^{-1}$$

and $K = \int_0^\infty K(\lambda) d\lambda$. Since ρ and $[D, \rho]$ have compact supports, it follows that $(\rho^2 + [D, \rho]\varepsilon)T^2$ and DKT^2 are both compact. This finishes the proof. \square

3.2 Finite propagation speed

In this subsection, we shall show that for any first order essentially selfadjoint differential operator $D : \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$, if the propagation speed of D is finite, that is,

$$c_D = \sup\{\|\sigma_D(x, \xi)\| : x \in X, \xi \in T_x^*X, \|\xi\| = 1\} < \infty,$$

then the unitary operators e^{isD} satisfy the following finite propagation property. The results in this subsection are straightforward generalizations of their corresponding classical results. We refer the reader to [12, Section 10.3] for detailed proofs.

In the rest of this subsection, let us fix a closed (not necessarily compact) subset $K \subset X$. We denote

$$Z_\beta = Z_\beta(K) = \{x \in X \mid d(x, K) < 2\beta\}$$

where $d(x, K)$ is the distance of x from K and $\beta > 0$ is some fixed constant.

Proposition 3.4. *Let*

$$D : \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$$

be a first order essentially self-adjoint differential operator on a complete Riemannian manifold X . Suppose D has finite propagation speed $c_D < \infty$. Then for all $\sigma \in \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$ supported within K , we have $e^{isD}\sigma$ supported in $Z_\beta(K)$, for all s with $|s| < \beta c_D^{-1}$.

Corollary 3.5. *Let φ be a bounded Borel function on \mathbb{R} whose Fourier transform is supported in $(-\beta c_D^{-1}, \beta c_D^{-1})$. If $\sigma \in \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$ is supported in K , then $\varphi(D)\sigma$ is supported in $Z_\beta(K)$.*

Proof. Since

$$\langle \varphi(D)\sigma, \nu \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle e^{isD}\sigma, \nu \rangle \widehat{\varphi}(s) ds,$$

the corollary follows immediately from the proposition above. \square

Corollary 3.6. *Let D_1 and D_2 be essentially selfadjoint differential operators on X which coincide on $Z_\beta(K)$. Suppose φ is a bounded Borel function on \mathbb{R} whose Fourier transform is supported in $(-\beta c_D^{-1}, \beta c_D^{-1})$. Then*

$$\varphi(D_1)\sigma = \varphi(D_2)\sigma$$

for all $\sigma \in \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$ supported in K .

3.3 Higher index classes

In this subsection, we construct the higher index class $\text{Ind}(D_\mathcal{V})$ (with finite propagation property) of the Dirac operator $D_\mathcal{V}$.

Let $D = D_\mathcal{V}$. A similar argument as that in Proposition 3.3 shows that

$$G = D(D^2 + \rho^2)^{-1/2}$$

is a generalized Fredholm operator. With respect to the \mathbb{Z}_2 grading,

$$G = \begin{pmatrix} 0 & U_G^* \\ U_G & 0 \end{pmatrix} = \begin{pmatrix} 0 & D^-(D^+D^- + \rho^2)^{-1/2} \\ D^+(D^-D^+ + \rho^2)^{-1/2} & 0 \end{pmatrix}.$$

Then the index of D is

$$\text{Ind}(D) := \text{Ind}(U_G) \in K_0(\mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)})) \cong K_0(C_r^*(\Gamma)).$$

Such a representative of the index class does not have finite propagation property in general. In the following discussion, we shall carry out an explicit construction to remedy this.

Before getting into the details, we would like to point out that one can also use the operator

$$F = D(D^2 + \rho^2 + [D, \rho]\varepsilon)^{-1/2}$$

to construct a representative of the index class $\text{Ind}(D)$. This requires some justification, since after all F is not an odd operator with respect to the \mathbb{Z}_2 -grading. In order to define an index class, we need to take the odd part of F . If we write

$$F = \begin{pmatrix} A & U_F^* \\ U_F & C \end{pmatrix},$$

then its odd part is

$$\begin{pmatrix} 0 & U_F^* \\ U_F & 0 \end{pmatrix}.$$

One readily verifies that

$$G - F = \int_0^\infty D(D^2 + \rho^2 + [D, \rho]\varepsilon + \lambda^2)^{-1}([D, \rho]\varepsilon)(D^2 + \rho^2 + \lambda^2)^{-1}d\lambda.$$

It follows that we can choose an appropriate ρ so that F is sufficiently close to G . In particular, U_F is sufficiently close to U_G . Since U_G is generalized Fredholm, it follows that U_F is also generalized Fredholm. Moreover,

$$\text{Ind}(D) = \text{Ind}(U_G) = \text{Ind}(U_F).$$

In fact, the operator

$$F' := (D + \varepsilon\rho)(D^2 + \rho^2 + [D, \rho]\varepsilon)^{-1/2}$$

also produces the same index class for D , since F' only differs from F by a compact operator $\varepsilon\rho(D^2 + \rho^2 + [D, \rho]\varepsilon)^{-1/2}$.

Notice that neither F nor F' produces an index class of finite propagation property yet. In the following, we shall approximate F' by an operator of finite propagation property and use the latter to construct a representative of the index class $\text{Ind}(D)$.

Definition 3.7. A smooth function $\chi : \mathbb{R} \rightarrow [-1, 1]$ is a normalizing function if

- (1) $\chi(-\lambda) = -\chi(\lambda)$ for all $\lambda \in \mathbb{R}$,
- (2) $\chi(\lambda) > 0$ for $\lambda > 0$,
- (3) $\chi(\lambda) \rightarrow \pm 1$ as $\lambda \rightarrow \pm\infty$.

Since $D + \varepsilon\rho$ is invertible, there exists a normalizing function χ such that

$$\begin{cases} \chi(\lambda) = 1 & \text{for } \lambda \geq a \\ \chi(\lambda) = -1 & \text{for } \lambda \leq -a \end{cases}$$

where $a > 0$ is a constant such that $\text{spec}(D + \varepsilon\rho) \cap (-a, a) = \emptyset$. In fact,

$$\chi(D + \varepsilon\rho) = F' = (D + \varepsilon\rho)(D^2 + \rho^2 + [D, \rho]\varepsilon)^{-1/2}.$$

Lemma 3.8. For any $\delta > 0$, there exists a normalizing function φ , for which its distributional Fourier transform $\widehat{\varphi}$ is compactly supported and for which $\xi\widehat{\varphi}(\xi)$ is a smooth function, such that

$$\sup_{\lambda \in \mathbb{R}} |\varphi(\lambda) - \chi(\lambda)| < \delta.$$

Proof. By our explicit choice of χ , we see that χ' has compact support. Therefore $\xi\widehat{\chi}(\xi) = \widehat{\chi}'(\xi)$ is a smooth function.

Let m be a smooth even function on \mathbb{R} whose Fourier transform is a compactly supported smooth function. Moreover, we assume that

$$\int_{\mathbb{R}} m(\lambda) d\lambda = 1.$$

Define $m_t(\lambda) = t^{-1}m(t^{-1}\lambda)$. It is easy to verify that $(m_t * \chi)$ is a normalizing function and its distributional Fourier transform

$$\widehat{m_t * \chi} = \widehat{m_t} \cdot \widehat{\chi}$$

is compactly supported. Moreover, $m_t * \chi \rightarrow \chi$ uniformly as $t \rightarrow 0$, since χ is uniformly continuous on \mathbb{R} . Now let $\varphi = m_t * \chi$ for some sufficiently small $t > 0$. Notice that $\xi\widehat{\varphi}(\xi)$ is a smooth function, since both $\widehat{m_t}$ and $\xi\widehat{\chi}(\xi)$ are smooth functions. This finishes the proof. \square

Definition 3.9. Define $F_\varphi = \varphi(D + \varepsilon\rho)$.

Since the distributional Fourier transform $\widehat{\varphi}$ of φ has compact support, it follows immediately from Corollary 3.5 that F_φ has finite propagation property. More precisely, we have the following lemma.

Lemma 3.10. *Suppose*

$$\text{supp}(\widehat{\varphi}) \subset (-b, b).$$

Let $\beta = c_D \cdot b$, where c_D is the propagation speed of $(D + \varepsilon\rho)$. Let K be a closed subset of X and

$$Z_\beta = \{x \in X \mid d(x, K) < 2\beta\}$$

where $d(x, K)$ is the distance of x from K . If $\sigma \in \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$ is supported within K , then $\varphi(D + \varepsilon\rho)\sigma$ is supported in Z_β .

We denote the odd-grading part of F_φ by $\begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$ and denote the odd-grading part of F' by $\begin{pmatrix} 0 & U_{F'}^* \\ U_{F'} & 0 \end{pmatrix}$, where $F' = \chi(D + \varepsilon\rho) = (D + \varepsilon\rho)(D^2 + \rho^2 + [D, \rho]\varepsilon)^{-1/2}$. By choosing φ sufficiently close to χ , we can make $\|F_\varphi - F'\|$ as small as we want. In particular, we can choose φ such that U is sufficiently close to $U_{F'}$.

To summarize, we have constructed an element U such that

- (a) U has finite propagation property;
- (b) U is a generalized Fredholm operator;
- (c) $\text{Ind}(D_V) = \text{Ind}(U)$.

Recall that, to construct the index class of the element $U \in \mathcal{B}(\mathcal{H}_{C_r^*(\Gamma)})$, we choose a element V such that $UV - 1$ and $VU - 1$ are in $\mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)})$. Then the idempotent

$$W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^{-1} = \begin{pmatrix} UV + UV(1 - UV) & (2 + UV)(1 - UV)U \\ V(1 - UV) & (1 - UV)^2 \end{pmatrix},$$

is a representative of the index class, where

$$W = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -V & 1 \end{pmatrix} \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

However, an arbitrary choice of V cannot guarantee that the resulting idempotent p still has finite propagation property. So to remedy this, we choose

$$V = U^*.$$

Then clearly

$$p = \begin{pmatrix} UU^* + UU^*(1 - UU^*) & (2 + UU^*)(1 - UU^*)U \\ U^*(1 - UU^*) & (1 - UU^*)^2 \end{pmatrix}$$

has finite propagation property. In general, $UU^* - 1$ and $U^*U - 1$ are not in $\mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)})$. As a result,

$$p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)}).$$

This is taken care of by the discussion in Section 2.2, since p is τ -close to $\mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)})$ (in the sense of Definition 2.1) when φ is sufficiently close to χ . Therefore, p defines a K -theory class in $K_0(\mathcal{K}(\mathcal{H}_{C_r^*(\Gamma)}))$. This class coincides with the index class $\text{Ind}(D_V)$.

Definition 3.11. We call the idempotent p constructed above an *idempotent of finite propagation* of the Dirac operator D_V .

Remark 3.12. To deal with manifolds of dimension $n \not\equiv 0 \pmod{8}$ in the real case, we work with Cl_n -linear Dirac operators, cf. [17, Section II.7]. Here Cl_n is the standard real Clifford algebra on \mathbb{R}^n with $e_i e_j + e_j e_i = -2\delta_{ij}$. We recall the definition of Cl_n -linear Dirac operators in the following. Consider the standard representation ℓ of Spin_n on Cl_n given by left multiplication. Let $P_{\text{spin}}(X)$ be the principal Spin_n -bundle of a n -dimensional spin manifold X , then we define \mathfrak{S} to be the vector bundle

$$\mathfrak{S} = P_{\text{spin}}(X) \times_{\ell} \text{Cl}_n.$$

Now let \mathcal{V} be a $C_r^*(\Gamma; \mathbb{R})$ -bundle over X as before. We denote the associated Dirac operator on $\mathfrak{S} \otimes \mathcal{V}$ by

$$\mathfrak{D} : \mathcal{L}^2(X, \mathfrak{S} \otimes \mathcal{V}) \rightarrow \mathcal{L}^2(X, \mathfrak{S} \otimes \mathcal{V}).$$

Notice that the right multiplication of Cl_n on \mathfrak{S} commutes with ℓ . So we see that \mathfrak{D} in fact defines a higher index class

$$\text{Ind}(\mathfrak{D}) \in \widehat{K}_0(C_r^*(\Gamma; \mathbb{R}) \widehat{\otimes} \text{Cl}_n) \cong \widehat{K}_n(C_r^*(\Gamma; \mathbb{R})) \cong K_n(C_r^*(\Gamma; \mathbb{R})).$$

Here \widehat{K}_* stands for \mathbb{Z}_2 -graded K -theory, $\widehat{\otimes}$ stands for \mathbb{Z}_2 -graded tensor product, cf. [14, Chapter III]. Notice that for a trivially graded C^* -algebra \mathcal{A} , we have $\widehat{K}_n(\mathcal{A}) \cong K_n(\mathcal{A})$.

This approach works equally well for the complex case, in which case K -theory takes periodicity 2 instead of 8.

Remark 3.13. In fact, there is a more geometric approach in the complex case. With the same notation as before, we assume X is an odd dimensional spin manifold. Then $\mathbb{R} \times X_{\Gamma}$ is a $(\mathbb{Z} \times \Gamma)$ -cover of $\mathbb{S}^1 \times X$, where \mathbb{S}^1 is the unit circle. Now define the corresponding $C_r^*(\mathbb{Z} \times \Gamma)$ -bundle over $\mathbb{S}^1 \times \Gamma$ by

$$\mathcal{W} = (\mathbb{R} \times X_{\Gamma}) \times_{\mathbb{Z} \times \Gamma} C_r^*(\mathbb{Z} \times \Gamma).$$

Then we have $\text{Ind}(D_{\mathcal{W}}) \in K_0(C_r^*(\mathbb{Z} \times \Gamma)) = K_0(C_r^*(\Gamma)) \oplus K_1(C_r^*(\Gamma))$. In fact, $\text{Ind}(D_{\mathcal{W}})$ lies in the second summand, that is, $\text{Ind}(D_{\mathcal{W}}) \in K_1(C_r^*(\Gamma))$.

4 A relative higher index theorem

In this section, we prove one of the main results, a *relative higher index theorem*, of the paper .

Let X_0 and X_1 be two even dimensional⁷ spin manifolds with complete Riemannian metrics of positive scalar curvature towards infinity. Assume

⁷In the real case, we assume $\dim X_0 = \dim X_1 \equiv 0 \pmod{8}$.

that we have compact subspaces $K_i \subset X_i$ such that there is an (orientation preserving) isometry $\Psi : \Omega_0 \rightarrow \Omega_1$, where $\Omega_i \subset X_i - K_i$ is a union of (not necessarily all) connected components of $X_i - K_i$ (see Figure 1). Let \mathcal{S}_i be the corresponding spinor bundle over X_i . We assume that Ψ lifts to a bundle isometry $\tilde{\Psi} : \mathcal{S}_0|_{\Omega_0} \rightarrow \mathcal{S}_1|_{\Omega_1}$.

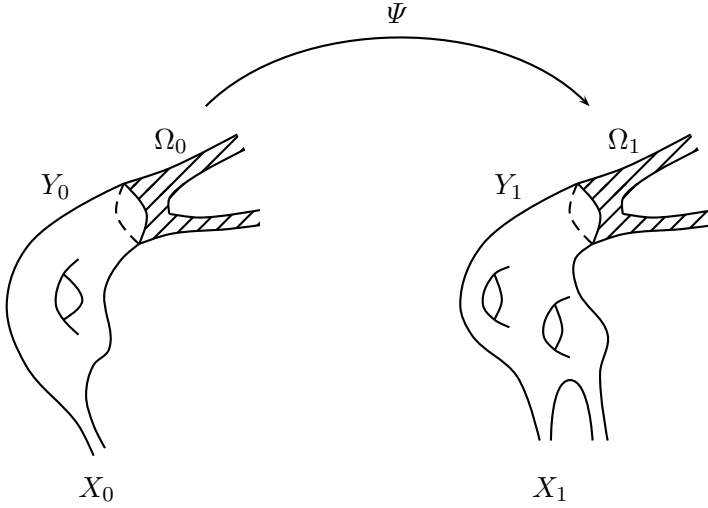


Figure 1: manifolds X_0 and X_1

Let $(X_i)_\Gamma$ be a Γ -cover of X_i , where Γ is a discrete group. Denote by

$$\pi_i : (X_i)_\Gamma \rightarrow X_i$$

the corresponding covering map. We assume that Ψ lifts to an isometry of the covers, also denoted by $\tilde{\Psi}$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \pi_0^{-1}(\Omega_0) & \xrightarrow{\tilde{\Psi}} & \pi_1^{-1}(\Omega_1) \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ \Omega_0 & \xrightarrow{\Psi} & \Omega_1 \end{array}$$

Define a flat $C_r^*(\Gamma)$ -bundle \mathcal{V}_i on X_i by

$$\mathcal{V}_i = (X_i)_\Gamma \times_\Gamma C_r^*(\Gamma).$$

Then $\tilde{\Psi}$ induces a bundle isometry from $\mathcal{V}_0|_{\Omega_0}$ to $\mathcal{V}_1|_{\Omega_1}$, which we still denote by $\tilde{\Psi}$. Notice that $\tilde{\Psi} : \mathcal{V}_0|_{\Omega_0} \rightarrow \mathcal{V}_1|_{\Omega_1}$ covers the isometry $\Psi : \Omega_0 \rightarrow \Omega_1$, that

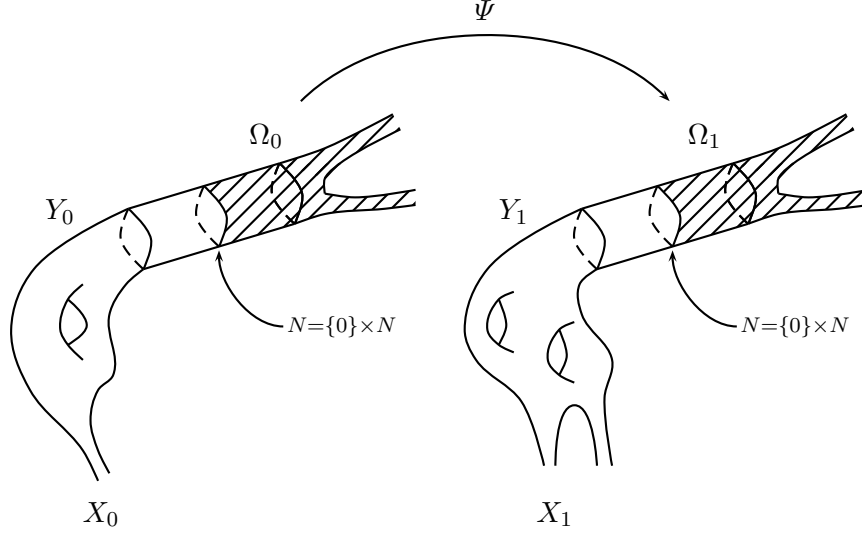


Figure 2: stretched manifolds X_0 and X_1

is, the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{V}_0|_{\Omega_0} & \xrightarrow{\tilde{\Psi}} & \mathcal{V}_1|_{\Omega_1} \\
 \downarrow & & \downarrow \\
 \Omega_0 & \xrightarrow{\Psi} & \Omega_1
 \end{array}$$

Let $D_i = D_{\mathcal{V}_i}$ be the associated Dirac operator on X_i , $i = 0, 1$. Then we have

$$D_1 = \tilde{\Psi} \circ D_0 \circ \tilde{\Psi}^{-1}$$

on Ω_1 . We say D_0 and D_1 agree on $\Omega = \Omega_0 \cong \Omega_1$.

Let N be a compact hypersurface in $\Omega \cong \Omega_i$ such that N cuts X_i into two components. We separate off the component that is inside Ω_i and denote the remaining part of X_i by Y_i (see Figure 1). Note that a deformation of the metric in a compact subset of a manifold does not affect the scalar curvature towards infinity, neither does it change the K -theory class of the higher index of the associated Dirac operator. So without loss of generality, we can assume Y_i has product metric in a small neighborhood of N .

In fact, in order to make use of the finite propagation property of our higher index classes, we further deform the metric near a collar neighborhood $(-\delta, \delta) \times N$ of N so that $(-\delta, \delta) \times N$ becomes $(-\ell, \ell) \times N$ for ℓ sufficiently large. Here we assume the standard Euclidean metric along the interval

$(-\ell, \ell)$. Now $N = \{0\} \times N$ cuts X_i into two components. We separate off the component that is inside Ω_i and denote the remaining part of X_i by $Y_i \cup ((-\ell, 0] \times N)$ (see Figure 2).

Remark 4.1. Recall that our choice of the normalizing function χ (Section 3.3) depends on the lower bound of $D_{\mathcal{V}} + \varepsilon\rho$. We claim that the stretching performed above does not affect the choice of χ . Indeed, since the stretching does not change the scalar curvature on the cylindrical neighborhood of N , it follows from the proof of Lemma 3.1 that the same lower bound works for the operator $D_{\mathcal{V}} + \varepsilon\rho$ before and after the stretching.

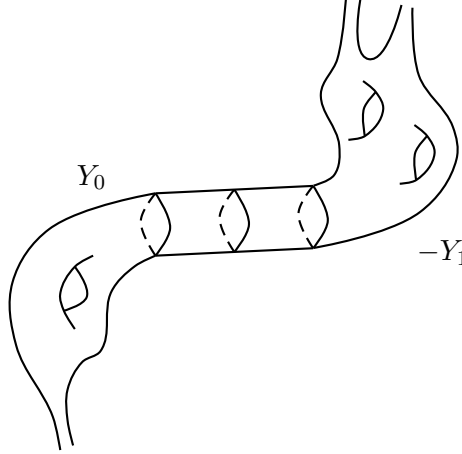


Figure 3: manifold X_2

Now we can glue $Y_0 \cup ((-\ell, 0] \times N)$ and $Y_1 \cup ((-\ell, 0] \times N)$ along $N = \{0\} \times N$. To do this, we need to reverse the orientation of one of the manifolds, say, $Y_1 \cup ((-\ell, 0] \times N)$. We denote the resulting manifold from this gluing by X_2 (see Figure 3). The spinor bundles \mathcal{S}_0 over $Y_0 \cup ((-\ell, 0] \times N)$ and \mathcal{S}_1 over $Y_1 \cup ((-\ell, 0] \times N)$ are glued together by the Clifford multiplication $c(v)$ to give a spinor bundle over X_2 , where $v = \frac{d}{du}$ is the inward unit normal vector near the boundary of $Y_0 \cup ((-\ell, 0] \times N)$. Moreover, the bundles $\mathcal{V}_0|_{Y_0}$ and $\mathcal{V}_1|_{Y_1}$ are glued together by $\tilde{\Psi}$ (near the boundary) to define a flat bundle \mathcal{V}_2 on X_2 . Let $D_2 = D_{\mathcal{V}_2}$ be the associated Dirac operator on X_2 .

Similarly, we can use two copies of Y_1 to construct a double of Y_1 . We define the manifold (see Figure 4)

$$X_3 = (Y_1 \cup ((-\ell, 0] \times N)) \bigcup_{\{0\} \times N} -(Y_1 \cup ((-\ell, 0] \times N))$$

and denote its associated Dirac operator by $D_3 = D_{\mathcal{V}_3}$.

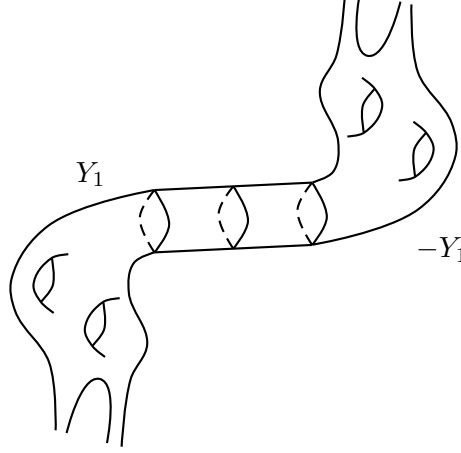


Figure 4: manifold X_3

We have the following relative higher index theorem.

Theorem 4.2.

$$\text{Ind}(D_2) = \text{Ind}(D_0) - \text{Ind}(D_1)$$

in $K_0(C_r^*(\Gamma))$.

Remark 4.3. If $\dim X = n$, then we have

$$\text{Ind}(D_2) = \text{Ind}(D_0) - \text{Ind}(D_1)$$

in $K_n(C_r^*(\Gamma))$ (resp. $K_n(C_r^*(\Gamma; \mathbb{R}))$ in the real case), cf. Remark 3.12.

Before we prove the theorem, let us fix some notation. Let p be an idempotent of finite propagation (in the sense of Definition 3.11) for D_0 and q an idempotent of finite propagation for D_1 . Since p and q have finite propagation property and the cylinder $(-\ell, \ell) \times N$ is sufficiently long (that is, ℓ is sufficiently large), we have

$$p(\sigma) = u^* q u(\sigma)$$

for all $\sigma \in \mathcal{L}^2(X_0 \setminus (Y_0 \cup ((-\ell, 0] \times N)), \mathcal{S}_0 \otimes \mathcal{V}_0)$, where

$$u : \mathcal{L}^2(X_0 \setminus (Y_0 \cup ((-\ell, 0] \times N)), \mathcal{S}_0 \otimes \mathcal{V}_0) \rightarrow \mathcal{L}^2(X_1 \setminus (Y_1 \cup ((-\ell, 0] \times N)), \mathcal{S}_1 \otimes \mathcal{V}_1)$$

is the unitary operator induced by the isometry $\tilde{\Psi} : \mathcal{S}_0 \otimes \mathcal{V}_0|_{\Omega_0} \rightarrow \mathcal{S}_0 \otimes \mathcal{V}_1|_{\Omega_1}$.

Definition 4.4. Define the following Hilbert modules over $C_r^*(\Gamma)$:

$$\begin{aligned}\mathcal{H}_1 &= \mathcal{L}^2(Y_0 \cup ((-\ell, 0] \times N), \mathcal{S}_0 \otimes \mathcal{V}_0), \\ \mathcal{H}_2 &= \mathcal{L}^2([0, \ell] \times N, \mathcal{S}_0 \otimes \mathcal{V}_0), \\ \mathcal{H}_3 &= \mathcal{L}^2(X_0 \setminus (Y_0 \cup ((-\ell, \ell] \times N)), \mathcal{S}_0 \otimes \mathcal{V}_0), \\ \mathcal{H}_4 &= \mathcal{L}^2(Y_1 \cup ((-\ell, 0] \times N), \mathcal{S}_1 \otimes \mathcal{V}_1).\end{aligned}$$

Notice that

$$\mathcal{L}^2(X_0, \mathcal{S}_0 \times \mathcal{V}_0) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

and

$$\mathcal{L}^2(X_1, \mathcal{S}_1 \times \mathcal{V}_1) = u(\mathcal{H}_2) \oplus u(\mathcal{H}_3) \oplus \mathcal{H}_4.$$

Let us denote

$$\tilde{\mathcal{H}}_{C_r^*(\Gamma)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4.$$

By finite propagation property of p and q , we have

$$p = \begin{pmatrix} p_{11} & p_{12} & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 \\ 0 & P_{32} & p_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & q_{32} & q_{33} & 0 \\ 0 & q_{42} & 0 & q_{44} \end{pmatrix}$$

in $\mathcal{B}(\tilde{\mathcal{H}}_{C_r^*(\Gamma)})$. Here $\mathcal{B}(\tilde{\mathcal{H}}_{C_r^*(\Gamma)})$ is the space of all adjointable operators in the Hilbert module $\tilde{\mathcal{H}}_{C_r^*(\Gamma)}$.

Proof of Theorem 4.2. Denote

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_1) \quad \text{and} \quad e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_4),$$

where the matrix form is used to denote the \mathbb{Z}_2 -grading of \mathcal{H}_2 and \mathcal{H}_3 . Define

$$\tilde{p} = \begin{pmatrix} p_{11} & p_{12} & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 \\ 0 & P_{32} & p_{33} & 0 \\ 0 & 0 & 0 & e_4 \end{pmatrix} \quad \text{and} \quad \tilde{q} = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ 0 & q_{22} & q_{23} & q_{24} \\ 0 & q_{32} & q_{33} & 0 \\ 0 & q_{42} & 0 & q_{44} \end{pmatrix}$$

Notice that $\tilde{p} - \tilde{q}$ is τ -close to $\mathcal{K}(\tilde{\mathcal{H}}_{C_r^*(\Gamma)})$ (in the sense of Definition 2.1). By applying the difference construction (cf. Section 2.3) to (\tilde{p}, \tilde{q}) , we obtain

$$\text{Ind}(D_0) - \text{Ind}(D_1) = E(\tilde{p}, \tilde{q}) \in K_0(\mathcal{K}(\tilde{\mathcal{H}}_{C_r^*(\Gamma)})) \cong K_0(C_r^*(\Gamma)).$$

We point out that, due to the presence of the term $\tilde{p} - \tilde{q}$ in all the nonzero entries in $E_0(\tilde{p}, \tilde{q})$ (cf. Formula (1) in Section 2.3), a straightforward calculation shows that the entries p_{23}, p_{32} and p_{33} (resp. q_{23}, q_{32} and q_{33}) in the matrix \tilde{p} (resp. \tilde{q}) do not appear in $E_0(\tilde{p}, \tilde{q})$. In other words, the summand \mathcal{H}_3 “disappears” when we pass to $E_0(\tilde{p}, \tilde{q})$.

Let p_1 (resp. q_1) be an idempotent of finite propagation for D_2 (resp. D_3). Similarly, define \tilde{p}_1 and \tilde{q}_1 as above, but in the Hilbert module

$$\tilde{\mathcal{H}}'_{C_r^*(\Gamma)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}'_3 \oplus \mathcal{H}_4$$

where $\mathcal{H}'_3 = \mathcal{L}^2(-Y_1, \mathcal{S}_1 \otimes \mathcal{V}_1)$. Note that

$$\mathcal{L}^2(X_2, \mathcal{S}_0 \times \mathcal{V}_0) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}'_3 \text{ and } \mathcal{L}^2(X_3, \mathcal{S}_0 \times \mathcal{V}_0) = \mathcal{H}_2 \oplus \mathcal{H}'_3 \oplus \mathcal{H}_4.$$

Then the difference construction gives

$$\text{Ind}(D_2) - \text{Ind}(D_3) = E(\tilde{p}_1, \tilde{q}_1) \in K_0(\mathcal{K}(\tilde{\mathcal{H}}'_{C_r^*(\Gamma)})) \cong K_0(C_r^*(\Gamma)).$$

Similarly, we see that the summand \mathcal{H}'_3 “disappears” when we pass to $E_0(\tilde{p}_1, \tilde{q}_1)$. In fact, we have $E_0(\tilde{p}_1, \tilde{q}_1) = E_0(\tilde{p}, \tilde{q})$ as matrices of operators in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_4)$. Therefore, we have

$$\text{Ind}(D_2) - \text{Ind}(D_3) = \text{Ind}(D_0) - \text{Ind}(D_1).$$

Now since D_3 is the associated Dirac operator over a double, it follows from Theorem 5.1 below that $\text{Ind}(D_3) = 0$. This finishes the proof. \square

5 Invertible Doubles

In this section, we carry out an invertible double construction (Theorem 5.1) for manifolds with (real or complex) C^* -vector bundles. This generalizes the invertible double construction for manifolds with classical vector bundles (i.e. \mathbb{C} -vector bundles or \mathbb{R} -vector bundles), cf. [7, Chapter 9]. For simplicity, we only state and prove the results for the complex case. The real case is proved by exactly the same argument.

Let Y_1 be an even dimensional complete manifold with boundary N , where N is a closed manifold. Assume that the Riemannian metric on Y_1 has positive scalar curvature towards infinity. Denote by \mathcal{S}_1 a Clifford bundle over Y_1 . Let \mathcal{V} be a \mathcal{A} -bundle over Y_1 , where \mathcal{A} is a C^* -algebra. Assume all metrics have product structures near the boundary. We denote a copy of

Y_1 with the reversed orientation by $Y_2 = -Y_1$ and denote the corresponding Clifford bundle by \mathcal{S}_2 . We glue Y_1 and Y_2 along a tubular neighborhood of the boundary to obtain a double \tilde{Y} of Y_1 . Now the bundles $\mathcal{S}_1 \otimes \mathcal{V}$ and $\mathcal{S}_2 \otimes \mathcal{V}$ are glued together by the Clifford multiplication $c(v)$, where $v = \frac{d}{du}$ is the inward unit normal vector near the boundary of Y_1 . We denote the resulting bundle on \tilde{Y} by $\tilde{\mathcal{S}} \otimes \tilde{\mathcal{V}}$. Note that

$$\tilde{\mathcal{S}}^\pm = \mathcal{S}_1^\pm \cup_{c(v)} \mathcal{S}_2^\mp.$$

In particular, a section of $\tilde{\mathcal{S}}^+ \otimes \tilde{\mathcal{V}}$ can be identified with a pair (s_1, s_2) such that s_1 is a section of $\mathcal{S}_1^+ \otimes \mathcal{V}$, s_2 is a section of $\mathcal{S}_2^- \otimes \mathcal{V}$ and near the boundary

$$s_2 = c(v)s_1.$$

Denote the Dirac operator over Y_i by

$$D_i^\pm : \Gamma(Y_i, \mathcal{S}_i^\pm \otimes \mathcal{V}) \rightarrow \Gamma(Y_i, \mathcal{S}_i^\mp \otimes \mathcal{V}).$$

Then the Dirac operator \tilde{D} on \tilde{Y} is identified with

$$\tilde{D}^\pm(s_1, s_2) = (D_1^\pm s_1, D_2^\mp s_2).$$

Theorem 5.1. *The operator \tilde{D} is bounded below, i.e., there exists a constant C such that*

$$\|\sigma\| \leq C \|\tilde{D}\sigma\|$$

for all $\sigma \in \Gamma^\infty(\tilde{Y}, \tilde{\mathcal{S}} \otimes \mathcal{V})$. In particular, the higher index class $\text{Ind}(\tilde{D})$ is zero.

Proof. Since each $\sigma \in \Gamma^\infty(\tilde{Y}, \tilde{\mathcal{S}}^+ \otimes \tilde{\mathcal{V}})$ can be identified with a pair (σ_1, σ_2) such that σ_1 is a section of $\mathcal{S}_1^+ \otimes \mathcal{V}$, σ_2 is a section of $\mathcal{S}_2^- \otimes \mathcal{V}$ and near the boundary

$$\sigma_2 = c(v)\sigma_1.$$

Therefore, by divergence theorem, we have

$$\int_{Y_1} \langle D^+ \sigma_1, \sigma_2 \rangle - \int_{Y_1} \langle \sigma_1, D^- \sigma_2 \rangle = - \int_{\partial Y_1} \langle c(v) \sigma_1, \sigma_2 \rangle = \int_N \langle \sigma_1, \sigma_1 \rangle. \quad (2)$$

It follows that there exists $k_1 > 0$ such that

$$\|\sigma|_N\|^2 = \int_N \langle \sigma_1, \sigma_1 \rangle \leq k_1 \|\tilde{D}^+ \sigma\| \|\sigma\|$$

Similarly, we have $\|\sigma|_N\|^2 \leq k_2 \|\tilde{D}^- \sigma\| \|\sigma\|$ for all $\sigma \in \Gamma^\infty(\tilde{Y}, \tilde{\mathcal{S}}^- \otimes \tilde{\mathcal{V}})$. Therefore, there exist a constant K_0 such that

$$\|\sigma|_N\|^2 \leq K_0 \|\tilde{D}\sigma\| \|\sigma\| \quad (3)$$

for all $\sigma \in \Gamma^\infty(\tilde{Y}, \tilde{\mathcal{S}} \otimes \tilde{\mathcal{V}})$.

Let $\Omega = (-\delta, \delta) \times N$ be a small tubular neighborhood of N in \tilde{Y} . Denote by $N_u = \{u\} \times N$ for $u \in (\delta, \delta)$. On the cylinder $(-\delta, \delta) \times N$, we have

$$\tilde{D} = c(u) \left(\frac{d}{du} + A \right)$$

where $c(u)$ is the Clifford multiplication of the normal direction $\frac{d}{du}$ and A is the Dirac operator on N . So we have a situation which is a special case of Lemma 5.11 below. It follows immediately from Lemma 5.11 that there exists a constant K_1 such that

$$\|\sigma|_\Omega\|^2 \leq K_1 \left(\|(\tilde{D}\sigma)|_\Omega\|^2 + \|\sigma|_N\|^2 \right)$$

for all $\sigma \in \Gamma^\infty(\tilde{Y}, \tilde{\mathcal{S}} \otimes \tilde{\mathcal{V}})$. Combined with the inequality (3), this implies that

$$\|\sigma|_\Omega\|^2 \leq K_2 (\|\tilde{D}\sigma\|^2 + \|\tilde{D}\sigma\| \|\sigma\|).$$

Now by the technical estimate (or rather its corollary 5.9) below, we have $\|\sigma\| \leq C_1 \|\sigma|_\Omega\| + C_2 \|\tilde{D}\sigma\|$, or equivalently,

$$\|\sigma\|^2 \leq C_1' \|\sigma|_\Omega\|^2 + C_2' \|\tilde{D}\sigma\|^2.$$

Therefore there exists a constant $C_0 > 0$ such that

$$\|\sigma\|^2 \leq C_0 (\|\tilde{D}\sigma\| \|\sigma\| + \|\tilde{D}\sigma\|^2)$$

i.e.

$$\frac{\|\tilde{D}\sigma\|}{\|\sigma\|} + \left(\frac{\|\tilde{D}\sigma\|}{\|\sigma\|} \right)^2 \geq \frac{1}{C_0}$$

for all nonzero $\sigma \in \Gamma^\infty(\tilde{Y}, \mathcal{S} \otimes \mathcal{V})$. So $\inf_{\sigma \neq 0} \frac{\|\tilde{D}\sigma\|}{\|\sigma\|} > 0$. This finishes the proof. \square

Remark 5.2. The above proof works for all dimensions, with obvious modifications by using Cl_n -linear Dirac operators as in Remark 3.12.

Remark 5.3. We point out that there is a natural (orientation reversing) reflection on the double \tilde{Y} . The reflection commutes with the Dirac operator \tilde{D} on \tilde{Y} . Using this, one sees that twice of the higher index class of \tilde{D} is zero, that is, $2\text{Ind}(\tilde{D}) = 0$ (for both the real and the complex cases).

Remark 5.4. We emphasize that the above proof works for both the real and the complex cases.

Remark 5.5. In the *complex* case, there is in fact a simpler way to show that $\text{Ind}(\tilde{D}) = 0$. We thank Ulrich Bunke for pointing this out to us. We provide the argument in the following. Note that however this argument does not work in the real case. Let \tilde{Y} be as above (of even dimension). Denote the grading operator on $\tilde{\mathcal{S}}$ by ε and the reflection on \tilde{Y} (and its induced action on $\tilde{\mathcal{S}} \otimes \tilde{\mathcal{V}}$) by J . Define $E = iJ\varepsilon$. Notice that J anticommutes with ε and commutes with \tilde{D} . So $E^2 = 1$ and $E\tilde{D} + \tilde{D}E = 0$. Then, since $(\tilde{D} + tE)^2 = \tilde{D}^2 + t^2$ is invertible when $t \in (0, 1]$, we see that $\tilde{D} + tE$ is invertible for all $t \in (0, 1]$. Therefore, $\text{Ind}(\tilde{D}) = \text{Ind}(\tilde{D} + tE) = 0$ by homotopy invariance of the index map.

5.1 A technical theorem

In this subsection, we prove the technical estimate that was used in the proof of Theorem 5.1.

First let us consider the case of compact manifolds. Let X be a compact Riemannian manifold and \mathcal{S} a $\text{Cl}(TX)$ -bundle with $\text{Cl}(TX)$ -compatible connection. Let \mathcal{A} be a C^* -algebra and let \mathcal{V} be a \mathcal{A} -bundle over X . Denote by D the associated generalized Dirac operator

$$D : \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V}).$$

Denote by $d(\cdot, \cdot)$ the Riemannian distance on X . Then for $\lambda > 0$, we define

$$\Omega^\lambda = \{x \in X \mid d(x, \Omega) < \lambda\}$$

for any open subset Ω of X . In the following, $\langle \cdot, \cdot \rangle$ stands for the \mathcal{A} -valued Hermitian product on \mathcal{V} and $\|\cdot\|$ denotes the L^2 -norm on $\mathcal{L}^2(X; \mathcal{S} \otimes \mathcal{V})$, unless otherwise specified.

Theorem 5.6. *With the above notation, fix an open subset Ω of X . Then there are constants C_1 and C_2 such that*

$$\|\sigma\| \leq C_1 \|\sigma|_\Omega\| + C_2 \|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$. Here $\sigma|_\Omega$ is the restriction of σ to Ω .

Proof. We reduce the theorem to the following claim.

Claim 5.7. *There exists a constant $\lambda > 0$ such that for any open subset $\Omega \subset X$, there exist constants $K_{\Omega,1}$ and $K_{\Omega,2}$ such that*

$$\|\sigma|_{\Omega^\lambda}\| \leq K_{\Omega,1}\|\sigma|_{\Omega}\| + K_{\Omega,2}\|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$.

Indeed, let λ be the constant from the claim. Denote $\Omega_1 = \Omega^\lambda$, then there are constants k_1 and k_2 such that

$$\|\sigma|_{\Omega_1}\| \leq k_1\|\sigma|_{\Omega}\| + k_2\|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$. Similarly, let $\Omega_2 = \Omega_1^\lambda$, then there are constants k_3 and k_4 such that

$$\|\sigma|_{\Omega_2}\| \leq k_3\|\sigma|_{\Omega_1}\| + k_4\|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$.

It follows immediately that

$$\begin{aligned} \|\sigma|_{\Omega_2}\| &\leq k_3\|\sigma|_{\Omega_1}\| + k_4\|D\sigma\| \\ &\leq k_4\|D\sigma\| + k_3(k_1\|\sigma|_{\Omega}\| + k_2\|D\sigma\|) \\ &= k_1k_3\|\sigma|_{\Omega}\| + (k_4 + k_2k_3)\|D\sigma\| \end{aligned}$$

Inductively, we define $\Omega_{k+1} = \Omega_k^\lambda$. Since X is compact, there exists an integer n such that $\Omega_n = X$. The theorem follows by a finite induction. \square

Remark 5.8. The constant λ is independent of the choice of Ω , although the constants $K_{\Omega,1}$, $K_{\Omega,2}$ and $K_{\Omega,3}$ may depend on Ω .

Now we shall generalize the above theorem to the case of complete manifolds with positive scalar curvature towards infinity.

Corollary 5.9. *Let X be a spin manifold with a complete Riemannian metric of positive scalar curvature towards infinity. Suppose Ω is an open subset of X with compact closure. Then there are constants C_1 and C_2 such that*

$$\|\sigma\| \leq C_1\|\sigma|_{\Omega}\| + C_2\|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$.

Proof. Since X has positive scalar curvature towards infinity, there exists an precompact open subset $\Sigma \subset X$ such that

- (1) $\Omega \subset \Sigma$,
- (2) the scalar curvature $\kappa \geq c_0 > 0$ on $X - \Sigma$.

The same argument as in the proof of Lemma 3.1 shows that there exists a constant c such that

$$\|D\sigma\| \geq \|(D\sigma)|_{X-\Sigma}\| \geq c\|\sigma|_{X-\Sigma}\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$. Now since the closure of Σ is compact, it follows from Theorem 5.6 that

$$\|\sigma|_\Sigma\| \leq c_1\|\sigma|_\Omega\| + c_2\|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$. Therefore we have

$$\|\sigma\| \leq \|\sigma|_{X-\Sigma}\| + \|\sigma|_\Sigma\| \leq C_1\|\sigma|_\Omega\| + C_2\|D\sigma\|.$$

□

5.2 Proof of Claim 5.7

In this subsection, we prove the claim in the proof of Theorem 5.6. Our argument is inspired by the proof of [7, Theorem 8.2].

With the same notation from the previous subsection, let $x_0 \in \partial\Omega$. Choose $r_0 > 0$ sufficiently small and $p \in \Omega$ at a distance r_0 from x_0 such that the ball $B(p_0; r_0)$ with center at p_0 and radius r_0 is contained in Ω . Choose spherical coordinates in a small neighborhood of p_0 . Denote the ball with center at p_0 and radius r by $B(p_0; r)$. See Figure 5 below.

Let $B = B(p_0; r_0)$ and ∂B its boundary. We define an \mathcal{A} -valued inner product

$$\langle\langle\sigma, \eta\rangle\rangle_s = \int_B \langle(1 + \Delta)^s \sigma(x), \eta(x)\rangle dx$$

for all $\sigma, \eta \in \mathcal{H}^s(B, \mathcal{S} \otimes \mathcal{V})$ (resp. for all $\sigma, \eta \in \mathcal{H}^s(\partial B, \mathcal{S} \otimes \mathcal{V})$), where Δ is the Laplacian operator on B (resp. ∂B).

Lemma 5.10. *For $k \geq 1$, we have*

$$\langle\langle\sigma|_{\partial B}, \sigma|_{\partial B}\rangle\rangle_{k-1/2} \leq C\langle\langle\sigma, \sigma\rangle\rangle_k$$

for all $\sigma \in \mathcal{H}^k(B, \mathcal{S} \otimes \mathcal{V})$.

Proof. See Appendix A. □

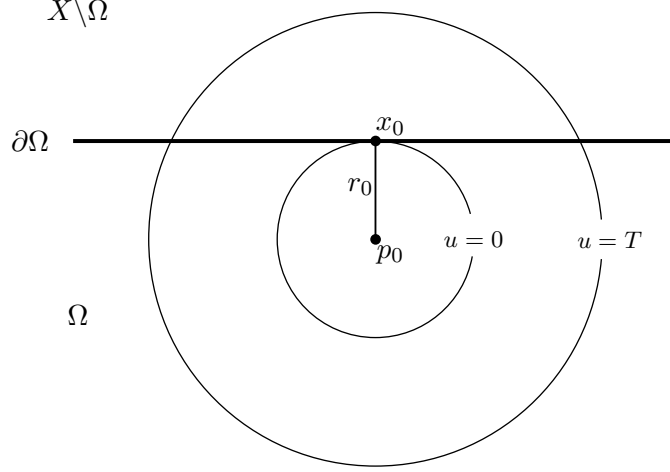


Figure 5: local spherical coordinates

The following lemma is a generalization of [7, Lemma 8.6]. In particular, our argument follows closely the proof of [7, Lemma 8.6].

Lemma 5.11. *For $R > 0$ sufficiently large and $T > 0$ sufficiently small, we have*

$$\begin{aligned} & R \int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle \sigma(u, y), \sigma(u, y) \rangle dy du \\ & \leq C \left(\int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle D\sigma, D\sigma \rangle dy du + RT e^{RT^2} \int_{\mathbb{S}_0} \langle \sigma, \sigma \rangle dy \right) \end{aligned} \quad (4)$$

for all $\sigma \in \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V})$, where \mathbb{S}_u is the sphere centered at p_0 with radius $r_0 + u$.

Proof. In order to simplify the computation, let us consider a few technical points. In the annulus $[0, T] \times \mathbb{S}_0$, the Dirac operator D takes the form

$$D = c(u) \left(\frac{\partial}{\partial u} + A_u \right)$$

where $c(u)$ is the Clifford multiplication of the radial vector. It is easy to see that we may consider the operator $\frac{\partial}{\partial u} + A_u$ instead of D . We may further deform the Riemannian metric of the manifold and the Hermitian metrics

of the bundles on $[0, T] \times \mathbb{S}_0$ such that they do not depend on the radial direction u , but *keep the operator D fixed*. Then it suffices to prove the lemma under this new metric. The only inconvenience is that in general A_u is not a self-adjoint operator with respect to the new structures. This is taken care of by considering its self-adjoint part

$$A_+ = (A_u)_+ = \frac{1}{2}(A_u + A_u^*).$$

Notice that $(A_u)_+$ is an elliptic differential operator for each u , as long as T is sufficiently small. A priori, the choice of T may depend on the neighborhood $B(p_0, r_0)$. However, since X is compact, we can choose $T > 0$ to be independent of $B(p_0, r_0)$.

Consider $\nu = e^{R(T-u)^2/2}\sigma$. Then the inequality (4) becomes

$$\begin{aligned} & R \int_{u=0}^T \int_{\mathbb{S}_u} \langle \nu(u, y), \nu(u, y) \rangle dy du \\ & \leq C \left(\iint \langle (D + R(T-u))\nu, (D + R(T-u))\nu \rangle dy du + RT \int_{\mathbb{S}_0} \langle \nu, \nu \rangle dy \right) \end{aligned} \quad (5)$$

Decompose $\frac{\partial}{\partial u} + A + R(T-u)$ into its symmetric part $A_+ + R(T-u)$ and its anti-symmetric part $\frac{\partial}{\partial u} + A_-$ with

$$A_- = (A_u)_- = \frac{1}{2}(A_u - A_u^*).$$

Then

$$\begin{aligned} & \int_{u=0}^T \int_{\mathbb{S}_u} \langle (D + R(T-u))\nu, (D + R(T-u))\nu \rangle dy du \\ & = \int_{u=0}^T \int_{\mathbb{S}_u} \left\langle \frac{\partial \nu}{\partial u} + A\nu + R(T-u)\nu, \frac{\partial \nu}{\partial u} + A\nu + R(T-u)\nu \right\rangle dy du \\ & = \iint \left\langle \frac{\partial \nu}{\partial u} + A_-\nu, \frac{\partial \nu}{\partial u} + A_-\nu \right\rangle dy du \\ & \quad + \iint \langle (A_+ + R(T-u))\nu, (A_+ + R(T-u))\nu \rangle dy du \\ & \quad + \iint \left\langle \frac{\partial \nu}{\partial u} + A_-\nu, A_+\nu + R(T-u)\nu \right\rangle dy du \\ & \quad + \iint \left\langle A_+\nu + R(T-u)\nu, \frac{\partial \nu}{\partial u} + A_-\nu \right\rangle dy du \end{aligned} \quad (6)$$

Let us consider the last two terms of (6). By integration by parts, we have

$$\begin{aligned}
& \iint \left\langle \frac{\partial \nu}{\partial u} + A_- \nu, A_+ \nu + R(T - u) \nu \right\rangle dy du \\
& + \iint \left\langle A_+ \nu + R(T - u) \nu, \frac{\partial \nu}{\partial u} + A_- \nu \right\rangle dy du \\
& = \iint \left\langle \frac{\partial \nu}{\partial u}, A_+ \nu + R(T - u) \nu \right\rangle dy du \\
& + \iint \left\langle A_+ \nu + R(T - u) \nu, \frac{\partial \nu}{\partial u} \right\rangle dy du \\
& + \iint \langle A_- \nu, A_+ \nu \rangle dy du + \iint \langle A_+ \nu, A_- \nu \rangle dy du \\
& = \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu + RT \nu \rangle dy \\
& - \iint \left\langle \nu, \left(\frac{\partial}{\partial u} (A_+ \nu + R(T - u) \nu) \right) \nu \right\rangle dy du \\
& + \iint \left\langle \nu, (A_+ + R(T - u)) \frac{\partial \nu}{\partial u} \right\rangle dy du + \iint \langle \nu, [A_+, A_-] \nu \rangle dy du \\
& = \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu + RT \nu \rangle dy \\
& + \iint \left\langle \nu, -\frac{\partial A_+}{\partial u} \nu + R \nu \right\rangle dy du + \iint \langle \nu, [A_+, A_-] \nu \rangle dy du \\
& = \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy - RT \int_{\mathbb{S}_0} \langle \nu, \nu \rangle dy \\
& + R \iint \langle \nu, \nu \rangle dy du + \iint \left\langle \nu, -\frac{\partial A_+}{\partial u} \nu + [A_+, A_-] \nu \right\rangle dy du. \tag{7}
\end{aligned}$$

Now we prove the lemma in three steps.

Step One: We shall prove that

$$\begin{aligned}
& \pm \int_0^T \int_{\mathbb{S}_u} \left\langle \nu, -\frac{\partial A_+}{\partial u} \nu + [A_+, A_-] \nu \right\rangle dy du \\
& \leq k \left(R \iint \langle \nu, \nu \rangle dy du \right. \\
& \quad \left. + \iint \langle (A_+ + R(T - u)) \nu, (A_+ + R(T - u)) \nu \rangle dy du \right) \tag{8}
\end{aligned}$$

for some constant $0 < k < 1$.

Notice that $a^*b + b^*a \leq \lambda a^*a + \frac{1}{\lambda} b^*b$ for all $a, b \in \mathcal{A}$ and for all $\lambda > 0$. Moreover, since the operators $(A_u)_+$ are first order elliptic operators, we have the following \mathcal{A} -valued Gårding's inequality (cf. Lemma A.2):

$$\langle\langle f, f \rangle\rangle_1 \leq c\langle\langle f, f \rangle\rangle_0 + c\langle\langle (A_u)_+f, (A_u)_+f \rangle\rangle_0$$

for all $f \in \Gamma^\infty(\mathbb{S}_u, \mathcal{S} \otimes \mathcal{V})$. Therefore, we have

$$\begin{aligned} & 2 \iint \left\langle \nu, -\frac{\partial A_+}{\partial u} \nu + [A_+, A_-] \nu \right\rangle dy du \\ & \leq \lambda \iint \langle \nu, \nu \rangle dy du \\ & \quad + \frac{1}{\lambda} \iint \left\langle \left(-\frac{\partial A_+}{\partial u} + [A_+, A_-]\right) \nu, \left(-\frac{\partial A_+}{\partial u} + [A_+, A_-]\right) \nu \right\rangle dy du \\ & \leq \lambda \iint \langle \nu, \nu \rangle dy du + \frac{c_1}{\lambda} \int \langle\langle \nu, \nu \rangle\rangle_1 du \\ & \leq \lambda \iint \langle \nu, \nu \rangle dy du + \frac{c_1 c}{\lambda} \int \langle\langle \nu, \nu \rangle\rangle_0 + \langle\langle A_+ \nu, A_+ \nu \rangle\rangle_0 du \\ & = \lambda \iint \langle \nu, \nu \rangle dy du + \frac{c_1 c}{\lambda} \left(\iint \langle \nu, \nu \rangle dy du + \iint \langle A_+ \nu, A_+ \nu \rangle dy du \right) \\ & \leq \left(\lambda + \frac{c_1 c}{\lambda} \right) \iint \langle \nu, \nu \rangle dy du + \frac{2c_1 c}{\lambda} \iint \langle R(T-u)\nu, R(T-u)\nu \rangle dy du \\ & \quad + \frac{2c_1 c}{\lambda} \iint \langle (A_+ + R(T-u))\nu, (A_+ + R(T-u))\nu \rangle dy du \\ & \leq \left(\lambda + \frac{c_1 c}{\lambda} (2R^2 T^2 + 1) \right) \iint \langle \nu, \nu \rangle dy du \\ & \quad + \frac{2c_1 c}{\lambda} \iint \langle (A_+ + R(T-u))\nu, (A_+ + R(T-u))\nu \rangle dy du. \end{aligned}$$

Choose $\lambda = R$, then

$$\begin{aligned} & \iint \left\langle \nu, -\frac{\partial A_+}{\partial u} \nu + [A_+, A_-] \nu \right\rangle dy du \\ & \leq R \left(\frac{1}{2} + c_1 c T^2 + \frac{c_1 c}{2R^2} \right) \iint \langle \nu, \nu \rangle dy du \\ & \quad + \frac{c_1 c}{R} \iint \langle (A_+ + R(T-u))\nu, (A_+ + R(T-u))\nu \rangle dy du. \end{aligned}$$

This proves (8) for R sufficiently large and T sufficiently small. Note that the constants c_1 and c depend on the local spherical coordinate chart. Since

the manifold X is compact, we see that c_1 and c are uniformly bounded on X . Therefore the choice of the constant T can be made independent of the local neighborhood $B(p_0, r_0)$.

Step Two. Now let us consider the term

$$\int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy.$$

Recall that $\nu = e^{R(T-u)^2/2} \sigma$. In particular, $\nu = \sigma$ on \mathbb{S}_T . It follows that

$$\begin{aligned} & \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy \\ &= \int_{\mathbb{S}_T} \langle \sigma, A_+ \sigma \rangle dy - \int_{\mathbb{S}_0} \langle \sigma, A_+ \sigma \rangle dy \\ &= \int_0^T \int_{\mathbb{S}_u} \frac{\partial}{\partial u} \langle \sigma, A_+ \sigma \rangle dy du \\ &= \int_0^T \int_{\mathbb{S}_u} \left\langle \frac{\partial \sigma}{\partial u}, A_+ \sigma \right\rangle dy du + \int_0^T \int_{\mathbb{S}_u} \left\langle \sigma, \frac{\partial A_+}{\partial u} \sigma \right\rangle dy du \\ &\quad + \int_0^T \int_{\mathbb{S}_u} \left\langle \sigma, A_+ \frac{\partial \sigma}{\partial u} \right\rangle dy du. \end{aligned}$$

Since A_+ is self-adjoint, we have

$$\int_0^T \int_{\mathbb{S}_u} \left\langle \sigma, A_+ \frac{\partial \sigma}{\partial u} \right\rangle dy du = \int_0^T \int_{\mathbb{S}_u} \left\langle A_+ \sigma, \frac{\partial \sigma}{\partial u} \right\rangle dy du.$$

By Lemma A.1 and Lemma A.2, we see that

$$\begin{aligned} & \int_0^T \int_{\mathbb{S}_u} \left\langle \frac{\partial \sigma}{\partial u}, A_+ \sigma \right\rangle dy du + \int_0^T \int_{\mathbb{S}_u} \left\langle A_+ \sigma, \frac{\partial \sigma}{\partial u} \right\rangle dy du \\ &\leq \int_0^T \int_{\mathbb{S}_u} \left\langle \frac{\partial \sigma}{\partial u}, \frac{\partial \sigma}{\partial u} \right\rangle dy du + \int_0^T \int_{\mathbb{S}_u} \langle A_+ \sigma, A_+ \sigma \rangle dy du \\ &\leq K_0 \int_0^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du + K_0 \int_0^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{S}_u} \left\langle \sigma, \frac{\partial A_+}{\partial u} \sigma \right\rangle dy du \\ &\leq K_1 \int_0^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du + K_1 \int_0^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du. \end{aligned}$$

It follows that

$$\begin{aligned} & \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy \\ & \leq K \int_0^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du + K \int_0^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du. \end{aligned}$$

In fact, the same argument shows that

$$\begin{aligned} & \pm \left(\int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy \right) \\ & \leq K \int_0^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du + K \int_0^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du. \end{aligned} \quad (9)$$

Step Three. Combining (6), (7) and (8) together, we have

$$\begin{aligned} & \int_{u=0}^T \int_{\mathbb{S}_u} \langle (D + R(T-u))\nu, (D + R(T-u))\nu \rangle dy du \\ & \geq \iint \left\langle \frac{\partial \nu}{\partial u} + A_- \nu, \frac{\partial \nu}{\partial u} + A_- \nu \right\rangle dy du \\ & \quad + (1-k) \iint \langle (A_+ + R(T-u))\nu, (A_+ + R(T-u))\nu \rangle dy du \\ & \quad + (1-k)R \iint \langle \nu, \nu \rangle dy du - RT \int_{\mathbb{S}_0} \langle \nu, \nu \rangle dy \\ & \quad + \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy. \end{aligned} \quad (10)$$

It follows that

$$\begin{aligned} & (1-k)R \int_0^T \int_{\mathbb{S}_u} \langle \nu, \nu \rangle dy du \\ & \leq \int_{u=0}^T \int_{\mathbb{S}_u} \langle (D + R(T-u))\nu, (D + R(T-u))\nu \rangle dy du \\ & \quad + RT \int_{\mathbb{S}_0} \langle \nu, \nu \rangle dy + \int_{\mathbb{S}_0} \langle \nu, A_+ \nu \rangle dy - \int_{\mathbb{S}_T} \langle \nu, A_+ \nu \rangle dy. \end{aligned}$$

Recall that $\nu = e^{R(T-u)^2/2} \sigma$. By applying (9) to the above inequality, we

have

$$\begin{aligned}
& (1-k)R \int_0^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle \sigma, \sigma \rangle dy du \\
& \leq \int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle D\sigma, D\sigma \rangle dy du + RT e^{RT^2} \int_{\mathbb{S}_0} \langle \sigma, \sigma \rangle dy \\
& \quad + K \int_0^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du + K \int_0^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du.
\end{aligned}$$

It follows immediately that

$$\begin{aligned}
& [(1-k)R - K] \int_0^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle \sigma, \sigma \rangle dy du \\
& \leq (1+K) \int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle D\sigma, D\sigma \rangle dy du + RT e^{RT^2} \int_{\mathbb{S}_0} \langle \sigma, \sigma \rangle dy.
\end{aligned}$$

The proof is finished by choosing R sufficiently large. \square

Now we use the above lemmas to prove Claim 5.7.

Proof of Claim 5.7. Recall that T in Lemma 5.11 can be chosen independent of the local small neighborhoods. Now since the boundary of Ω is compact, then for all sufficiently small λ , we have

$$\Omega^\lambda \subset \Omega \cup \bigcup_{i=0}^N B(p_i; r_i + T)$$

for some $N \in \mathbb{N}$. Therefore, it suffices to show that we have constants C_1 and C_2 such that

$$\|\sigma|_{\Omega'}\| \leq C_1 \|\sigma|_{\Omega}\| + C_2 \|D\sigma\|$$

for all $\sigma \in \Gamma^\infty(X; \mathcal{S} \otimes \mathcal{V})$, where $\Omega' = \Omega \cup B(p_0; r_0 + T)$.

By Lemma 5.11, we have

$$\begin{aligned}
& \int_{\Omega'} \langle \sigma, \sigma \rangle dg \leq \int_{\Omega} \langle \sigma, \sigma \rangle dg + \int_{B(p_0; r_0+T)} \langle \sigma, \sigma \rangle dg \\
& = \int_{\Omega} \langle \sigma, \sigma \rangle dg + \int_{B(p_0; r_0)} \langle \sigma, \sigma \rangle dg + \int_{u=0}^T \int_{\mathbb{S}_u} \langle \sigma, \sigma \rangle dy du \\
& \leq 2 \int_{\Omega} \langle \sigma, \sigma \rangle dg + \int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle \sigma, \sigma \rangle dy du
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_{\Omega} \langle \sigma, \sigma \rangle dg \\
&\quad + \frac{C}{R} \left(\int_{u=0}^T \int_{\mathbb{S}_u} e^{R(T-u)^2} \langle D\sigma, D\sigma \rangle dy du + RT e^{RT^2} \int_{\mathbb{S}_0} \langle \sigma, \sigma \rangle dy \right) \\
&\leq 2 \int_{\Omega} \langle \sigma, \sigma \rangle dg + \frac{C e^{RT^2}}{R} \left(\int_{u=0}^T \int_{\mathbb{S}_u} \langle D\sigma, D\sigma \rangle dy du + RT \int_{\mathbb{S}_0} \langle \sigma, \sigma \rangle dy \right)
\end{aligned}$$

for R sufficiently large. Here dg stands for the volume form on X . Now by Lemma 5.10, it follows that

$$\|\sigma|_{\Omega'}\|^2 \leq K_1 \|\sigma|_{\Omega}\|^2 + K_2 \|D\sigma\|^2$$

for some constants K_1 and K_2 . This finishes the proof. \square

6 Diffeomorphisms and positive scalar curvature

In this section, we apply our relative higher index theorem to study $\mathcal{R}^+(M)$ the space of all metrics of positive scalar curvature on a manifold M . All the results and their proofs in this section work for both the real and the complex cases. For simplicity, we only state and prove the results for the complex case.

Throughout this section, we assume that M is an odd dimensional⁸ closed spin manifold and M_{Γ} a Γ -cover of M , where Γ is a discrete group. Assume M carries positive scalar curvature, i.e. $\mathcal{R}^+(M) \neq \emptyset$. Choose $g_0, g_1 \in \mathcal{R}^+(M)$. We define a smooth path of Riemannian metrics g_t on M such that

$$g_t = \begin{cases} g_0 & \text{for } t \leq 0, \\ g_1 & \text{for } t \geq 1, \\ \text{any smooth homotopy from } g_0 \text{ to } g_1 & \text{for } 0 \leq t \leq 1. \end{cases}$$

Then $X = M \times \mathbb{R}$ endowed with the metric $h = g_t + (dt)^2$ becomes a complete Riemannian manifold with positive scalar curvature towards infinity.

Denote $X_{\Gamma} = M_{\Gamma} \times \mathbb{R}$. Then X_{Γ} is naturally a Γ -cover of X with Γ acting on \mathbb{R} trivially. We define a flat $C_r^*(\Gamma)$ -bundle \mathcal{V} on X by

$$\mathcal{V} = X_{\Gamma} \times_{\Gamma} C_r^*(\Gamma).$$

⁸In the real case, assume that $\dim M = -1 \pmod{8}$.

Let $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the spinor bundle over X . Then, with the flat connection on \mathcal{V} , we can define the Dirac operator

$$D_{\mathcal{V}} : \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{S} \otimes \mathcal{V}). \quad (11)$$

By the discussion in Section 3, we have a higher index class $\text{Ind}(D_{\mathcal{V}}) \in K_0(C_r^*(\Gamma))$. We also write $\text{Ind}(D_{\mathcal{V}}) = \text{Ind}_\Gamma(g_0, g_1)$ if we want to specify the metrics.

Open Question. *It remains an open question whether $\text{Ind}_\Gamma(g_0, g_1)$ lies in the image of the Baum-Connes assembly map $\mu : K_0^\Gamma(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma))$.*

We refer the reader to [3] and [4] for a detailed description of the Baum-Connes assembly map ([5] for its real analogue).

Now let $g_0, g_1, g_2 \in \mathcal{R}^+(M)$ be Riemannian metrics of positive scalar curvature on M . The following propositions generalize the corresponding classical results of Gromov and Lawson [11, Theorem 4.41 & Theorem 4.48].

Proposition 6.1.

$$\text{Ind}_\Gamma(g_0, g_1) + \text{Ind}_\Gamma(g_1, g_2) = \text{Ind}_\Gamma(g_0, g_2).$$

Proof. The statement follows immediately from the relative higher index theorem (Theorem 4.2). \square

Denote by $\text{Diff}^\infty(M)$ the group of diffeomorphisms on M . For a fixed metric $g \in \mathcal{R}^+(M)$, set

$$\text{Ind}_\Gamma(\Psi) = \text{Ind}_\Gamma(g, (\Psi^{-1})^*g)$$

for $\Psi \in \text{Diff}^\infty(M)$.

Recall that in the case when $\Gamma = \pi_1(M)$ the fundamental group of M , there is a natural homomorphism

$$\varphi : \text{MCG}(M) = \text{Diff}^\infty(M)/\text{Diff}_0^\infty(M) \rightarrow \text{Out}(\Gamma)$$

where $\text{Out}(\Gamma)$ is the group of outer automorphisms of Γ and $\text{Diff}_0^\infty(M)$ is the connected component of the identity in $\text{Diff}^\infty(M)$. In particular, each $\Psi \in \text{Diff}^\infty(M)$ induces an automorphism $\Psi_* : K_0(C_r^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma))$. We denote by $K_0(C_r^*(\Gamma)) \rtimes_\varphi \text{MCG}(M)$ the semi-direct product of $K_0(C_r^*(\Gamma))$ and $\text{MCG}(M)$, where $\text{MCG}(M)$ acts on $K_0(C_r^*(\Gamma))$ through φ .

Proposition 6.2. *For $\Gamma = \pi_1(M)$, we have a group homomorphism*

$$\text{Ind}_\Gamma : \text{Diff}^\infty(M)/\text{Diff}_0^\infty(M) \rightarrow K_0(C_r^*(\Gamma)) \rtimes_\varphi \text{MCG}(M).$$

Proof. Note that

$$\Psi_*[\text{Ind}_\Gamma(g_0, g_1)] = \text{Ind}_\Gamma((\Psi^{-1})^*g_0, (\Psi^{-1})^*g_1)$$

for all $\Psi \in \text{Diff}^\infty(M)$. It follows that

$$\begin{aligned} \text{Ind}_\Gamma(\Psi_1 \circ \Psi_2) &= \text{Ind}_\Gamma(g, (\Psi_1^{-1})^* \circ (\Psi_2^{-1})^*g) \\ &= \text{Ind}_\Gamma(g, (\Psi_1^{-1})^*g) + \text{Ind}_\Gamma((\Psi_1^{-1})^*g, (\Psi_1^{-1})^* \circ (\Psi_2^{-1})^*g) \\ &= \text{Ind}_\Gamma(g, (\Psi_1^{-1})^*g) + \Psi_*[\text{Ind}_\Gamma(g, (\Psi_2^{-1})^*g)]. \end{aligned}$$

Clearly, the map Ind_Γ is trivial on $\text{Diff}_0^\infty(M)$. Hence follows the lemma. \square

6.1 Applications

For the rest of the section, we fix a Riemannian metric $g_0 \in \mathcal{R}^+(M)$ and fix a spin structure on M . Let $\Psi \in \text{Diff}^\infty(M)$. Assume that Ψ preserves the orientation of M and the spin structure on M . Let $g_1 = (\Psi^{-1})^*g_0$.

From now on, let $\Gamma = \pi_1(M)$. Consider the mapping cylinder $M_\Psi = (M \times [0, 1]) / \sim$, where \sim is the equivalence relation $(x, 0) \sim (\Psi(x), 1)$ for $x \in M$. Now $M_\Gamma = \widetilde{M}$ is the universal cover of M . Note that Ψ induces an outer automorphism $\Psi_* \in \text{Out}(\Gamma)$. More precisely, $\Psi_* : \Gamma \rightarrow \Gamma$ is only well defined modulo inner automorphisms. In the following, we fix a representative in the class of this outer automorphism. We shall see that our results below, which are stated at the level of K -theory, do not depend on such a choice.

Let $\Gamma \rtimes_\Psi \mathbb{Z}$ be the semi-direct product with the action \mathbb{Z} on Γ induced by Ψ . We shall simply write $\Gamma \rtimes \mathbb{Z}$ for $\Gamma \rtimes_\Psi \mathbb{Z}$ if no ambiguity arises. We see that $M_\Gamma \times \mathbb{R}$ is a $(\Gamma \rtimes \mathbb{Z})$ -cover of M_Ψ .

Consider the natural inclusion $\iota : \Gamma \hookrightarrow \Gamma \rtimes \mathbb{Z}$. We denote the induced inclusion map on C^* -algebras also by $\iota : C_r^*(\Gamma) \rightarrow C_r^*(\Gamma \rtimes \mathbb{Z})$. Then we have a homomorphism

$$\iota_* : K_0(C_r^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z})).$$

Note that an inner automorphism of Γ induces an inner automorphism of $C_r^*(\Gamma \rtimes \mathbb{Z})$, hence its induced automorphism on $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$ is the identity map. It follows that ι_* is independent of the choice of the representative for the outer automorphism class $\Psi_* \in \text{Out}(\Gamma)$.

Before we prove the main result of this section, let us fix some notation. Let g_t be a smooth path of Riemannian metrics on M such that

$$g_t = \begin{cases} g_0 & \text{for } t \leq 0, \\ g_1 = (\Psi^{-1})^*g_0 & \text{for } t \geq 1, \\ \text{any smooth homotopy from } g_0 \text{ to } g_1 & \text{for } 0 \leq t \leq 1. \end{cases}$$

We consider the following list of Dirac operators and their index classes.

- (a) For the manifold $X = M \times \mathbb{R}$ with the Riemannian metric $h = g_t + (dt)^2$, we denote by $D_{\mathcal{V}}$ its Dirac operator with coefficients in $\mathcal{V} = X_{\Gamma} \times_{\Gamma} C_r^*(\Gamma)$.
- (b) We denote the same manifold $M \times \mathbb{R}$ but with the metric $(\Psi^{-n})^*h$ by X_n . Let $\mathcal{V}_n = \mathcal{V}$ be the corresponding flat $C_r^*(\Gamma)$ -bundle over X_n . Then \mathbb{Z} acts isometrically on the disjoint union $X_{\mathbb{Z}} = \bigcup_{n \in \mathbb{Z}} X_n$ by

$$n \mapsto \Psi^n : X_k \rightarrow X_{k+n}.$$

The action of \mathbb{Z} actually lifts to an action on $\mathcal{V}_{\mathbb{Z}} = \bigcup_{n \in \mathbb{Z}} \mathcal{V}_n$. Equivalently, we consider the following flat $C_r^*(\Gamma \rtimes \mathbb{Z})$ -bundle over X :

$$\mathcal{W} = \mathcal{V}_{\mathbb{Z}} \times_{\mathbb{Z}} C_r^*(\mathbb{Z}).$$

With the metric h on X , we denote by $D_0 = D_{\mathcal{W}}$ the associated Dirac operator on X with coefficients in \mathcal{W} . Then

$$\iota_*(\text{Ind}(D_{\mathcal{V}})) = \text{Ind}(D_0)$$

in $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$.

- (c) We denote the manifold $M \times \mathbb{R}$ but with the product metric $g_0 + (dt)^2$ by X' . Then the same construction from (b) produces a Dirac operator $D_1 = D_{\mathcal{W}'}$ on X' . Since X' can be viewed as a double, it follows from Theorem 5.1 that $\text{Ind}(D_1) = 0$ in $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$. In fact, one can directly show that $\text{Ind}(D_1) = 0$ without referring to Theorem 5.1. Indeed, since g_0 has positive scalar curvature, $g_0 + (dt)^2$ also has positive scalar curvature everywhere on X' . This immediately implies that D_1 is bounded below. Therefore $\text{Ind}(D_1) = 0$.
- (d) Let \mathcal{S}_{Ψ} be the spinor bundle over the mapping cylinder M_{Ψ} . Define a flat $C_r^*(\Gamma \rtimes \mathbb{Z})$ -bundle over M_{Ψ} by

$$\mathcal{W}_{\Psi} = (M_{\Gamma} \times \mathbb{R}) \times_{\Gamma \rtimes \mathbb{Z}} C_r^*(\Gamma \rtimes \mathbb{Z}).$$

We denote the associated Dirac operator by $D_2 = D_{\mathcal{W}_{\Psi}}$. Note that, since M_{Ψ} is closed, $\text{Ind}(D_2)$ does not depend on which Riemannian metric we have on M_{Ψ} . Equivalently, we view $M_{\Gamma} \times \mathbb{R}$ as a $\Gamma \rtimes \mathbb{Z}$ -cover of M_{Ψ} . Then the Dirac operator $D_{\Gamma \rtimes \mathbb{Z}}$ on $M_{\Gamma} \times \mathbb{R}$ defines a higher index class $\text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$ in $K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$ (cf. [9, Section 5]). By construction, we have $\text{Ind}(D_2) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$.

Recall that we write $\text{Ind}(D_\Psi) = \text{Ind}_\Gamma(\Psi)$. The following theorem provides a formula in many cases to determine when $\iota_*(\text{Ind}_\Gamma(\Psi))$ is nonvanishing (e.g. when the strong Novikov conjecture holds for $\Gamma \rtimes \mathbb{Z}$).

Theorem 6.3. *With the above notation, we have*

$$\iota_*(\text{Ind}_\Gamma(\Psi)) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$$

in $K_0(C_r^(\Gamma \rtimes \mathbb{Z}))$. In particular, this implies that $\iota_*(\text{Ind}_\Gamma(\Psi))$ lies in the image of the Baum-Connes assembly map*

$$\mu : K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z})).$$

Remark 6.4. If $m = \dim M \neq -1 \pmod{8}$ in the real case (resp. if m is even in the complex case), we consider the Cl_n -linear Dirac operator as in Remark 3.12. The same proof below implies that $\iota_*(\text{Ind}_\Gamma(\Psi)) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$ in $K_{m+1}(C_r^*(\Gamma \rtimes \mathbb{Z}; \mathbb{R}))$ (resp. $K_1(C_r^*(\Gamma \rtimes \mathbb{Z}))$). Again, in the complex case, one can in fact apply the above theorem to $\mathbb{S}^1 \times M$ to cover the case of even dimensional manifolds.

Proof of Theorem 6.3. Notice that the left end Ω'_l (resp. the right end Ω'_r) of X' is isometric to the left end Ω_l (resp. the right end Ω_r) of X through the identity map (resp. the diffeomorphism Ψ). Moreover, the isometries lift to isometries from $\mathcal{W}|_{\Omega'_l}$ to $\mathcal{W}|_{\Omega_l}$ (resp. from $\mathcal{W}|_{\Omega'_r}$ to $\mathcal{W}|_{\Omega_r}$). So we can apply our relative higher index theorem (Theorem 4.2) to D_0 and D_1 , where we identify $\Omega_0 = \Omega'_l \cup \Omega'_r$ with $\Omega_1 = \Omega_l \cup \Omega_r$.

Following the cutting-pasting procedure in Section 4, we see that X' and X join together to give exactly M_Ψ . Moreover, the $C_r^*(\Gamma \rtimes \mathbb{Z})$ -bundles \mathcal{W}' and \mathcal{W} join together to give precisely the bundle \mathcal{W}_Ψ over M_Ψ . Therefore, by Theorem 4.2, we have

$$\text{Ind}(D_2) = \text{Ind}(D_0) - \text{Ind}(D_1).$$

By the discussion above, we have $\text{Ind}(D_0) = \iota_*(\text{Ind}_\Gamma(\Psi))$, $\text{Ind}(D_1) = 0$ and $\text{Ind}(D_2) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$. This finishes the proof. \square

Open Question. *It remains an open question whether $\text{Ind}_\Gamma(\Psi)$ lies in the image of the Baum-Connes assembly map*

$$\mu : K_0^\Gamma(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma)).$$

In the following, we show that this question has an affirmative answer for some special cases.

Recall that Ψ induces an automorphism $\Psi_* : \Gamma \rightarrow \Gamma$ (up to inner automorphisms), thus an automorphism $\Psi_* : C_r^*(\Gamma) \rightarrow C_r^*(\Gamma)$ (up to inner automorphisms). Therefore, we have well-defined isomorphisms

$$\Psi_* : K_i(C_r^*(\Gamma)) \rightarrow K_i(C_r^*(\Gamma)), \quad i = 0, 1.$$

Corollary 6.5. *If $\Psi_* = \text{Id} : \Gamma \rightarrow \Gamma$, then $\text{Ind}_\Gamma(\Psi)$ lies in the image of the Baum-Connes assembly map $\mu : K_0^\Gamma(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma))$.*

Proof. Since $\Psi_* = \text{Id} : \Gamma \rightarrow \Gamma$, it follows immediately that

$$C_r^*(\Gamma \rtimes \mathbb{Z}) \cong C_r^*(\Gamma) \otimes C_r^*(\mathbb{Z}).$$

Then we have

$$K_0(C_r^*(\Gamma \rtimes \mathbb{Z})) \cong K_0(C_r^*(\Gamma)) \oplus K_1(C_r^*(\Gamma)).$$

Similarly, $K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \cong K_0^\Gamma(\underline{E}\Gamma) \oplus K_1^\Gamma(\underline{E}\Gamma)$. Moreover, the Baum-Connes assembly map respects this direct sum decomposition. Now the map

$$\iota_* : K_0(C_r^*(\Gamma)) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z})) \cong K_0(C_r^*(\Gamma)) \oplus K_1(C_r^*(\Gamma))$$

is simply $[p] \mapsto ([p], [0])$. Since $\iota_*(\text{Ind}_\Gamma(\Psi))$ lies in the image of the Baum-Connes assembly map $\mu : K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$, it follows that $\text{Ind}_\Gamma(\Psi)$ lies in the image of the Baum-Connes assembly map

$$\mu : K_0^\Gamma(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma)).$$

□

Corollary 6.6. *Assume that $\Psi_* = \text{Id} : K_i(C_r^*(\Gamma)) \rightarrow K_i(C_r^*(\Gamma))$ and in addition that the strong Novikov conjecture holds for Γ . Then $\text{Ind}_\Gamma(\Psi)$ lies in the image of the Baum-Connes assembly map $\mu : K_0^\Gamma(\underline{E}\Gamma) \rightarrow K_0(C_r^*(\Gamma))$.*

Proof. By Pimsner-Voiculescu exact sequence⁹, we have

$$\begin{array}{ccccc} K_0(C_r^*(\Gamma)) & \xrightarrow{1-\Psi_*} & K_0(C_r^*(\Gamma)) & \xrightarrow{\iota_*} & K_0(C_r^*(\Gamma \rtimes \mathbb{Z})) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(C_r^*(\Gamma \rtimes \mathbb{Z})) & \xleftarrow{\iota_*} & K_1(C_r^*(\Gamma)) & \xleftarrow{1-\Psi_*} & K_1(C_r^*(\Gamma)) \end{array}$$

⁹In the real case, the Pimsner-Voiculescu exact sequence has 24 terms instead.

Similarly, we have the six-term exact sequence

$$\begin{array}{ccccc}
K_0^\Gamma(\underline{E}\Gamma) & \xrightarrow{1-\Psi_*} & K_0^\Gamma(\underline{E}\Gamma) & \xrightarrow{\iota_*} & K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \\
\partial_1 \uparrow & & & & \downarrow \partial_0 \\
K_1^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) & \xleftarrow{\iota_*} & K_1^\Gamma(\underline{E}\Gamma) & \xleftarrow{1-\Psi_*} & K_1^\Gamma(\underline{E}\Gamma)
\end{array}$$

Moreover, the Baum-Connes assembly map is natural with respect to these exact sequences. So by our assumption that $\Psi_* = \text{Id} : K_i(C_r^*(\Gamma)) \rightarrow K_i(C_r^*(\Gamma))$, we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_0^\Gamma(\underline{E}\Gamma) & \xrightarrow{\iota_*} & K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) & \xrightarrow{\partial_0} & K_1^\Gamma(\underline{E}\Gamma) \longrightarrow 0 \\
& & \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
0 & \longrightarrow & K_0(C_r^*(\Gamma)) & \xrightarrow{\iota_*} & K_0(C_r^*(\Gamma \rtimes \mathbb{Z})) & \xrightarrow{\partial_0} & K_1(C_r^*(\Gamma)) \longrightarrow 0
\end{array}$$

By Theorem 6.3 above, we have $\iota_*(\text{Ind}_\Gamma(\Psi)) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$. Since $\text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$ lies in the image of the Baum-Connes assembly map $\mu : K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z})) \rightarrow K_0(C_r^*(\Gamma \rtimes \mathbb{Z}))$, there is an element $P \in K_0^{\Gamma \rtimes \mathbb{Z}}(\underline{E}(\Gamma \rtimes \mathbb{Z}))$ such that $\mu(P) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}})$. Notice that $\mu \circ \partial_0(P) = \partial_0 \circ \mu(P) = 0$. Since the strong Novikov conjecture holds for Γ , that is, the map $\mu : K_i^\Gamma(\underline{E}\Gamma) \rightarrow K_i(C_r^*(\Gamma))$ is injective, it follows that $\partial_0(P) = 0$. Therefore, there exist an element $Q \in K_0^\Gamma(\underline{E}\Gamma)$ such that $\iota_*(Q) = P$. The following diagram shows how all these elements are related under various maps:

$$\begin{array}{ccccc}
Q & \xrightarrow{\iota_*} & P & \xrightarrow{\partial_0} & \partial_0(P) \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
\text{Ind}_\Gamma(\Psi) & \xrightarrow{\iota_*} & \iota_*(\text{Ind}_\Gamma(\Psi)) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}}) & \xrightarrow{\partial_0} & 0
\end{array}$$

In particular, we see that $\iota_* \circ \mu(Q) = \mu \circ \iota_*(P) = \text{Ind}(D_{\Gamma \rtimes \mathbb{Z}}) = \iota_*(\text{Ind}_\Gamma(\Psi))$. Now since $\Psi_* = \text{Id}$, it follows immediately that $\mu(Q) = \text{Ind}_\Gamma(\Psi)$. This finishes the proof. \square

Remark 6.7. The above corollaries have their counterparts for all other dimensions and for both the real and the complex cases, which are essentially proved by the same arguments as above.

A Technical lemmas

In this appendix, we prove some standard estimates for pseudodifferential operators with coefficients in \mathcal{A} -bundles, where \mathcal{A} is an arbitrary real or complex C^* -algebra. In particular, we prove Gårding's inequality in this setting. We would like to point out that all the estimates take values in \mathcal{A} rather than \mathbb{R} (or \mathbb{C}).

Let X be a compact Riemannian manifold. Let \mathcal{A} be a C^* -algebra and \mathcal{V} an \mathcal{A} -bundle over X . We denote the \mathcal{A} -valued inner product on \mathcal{V} by $\langle \cdot, \cdot \rangle$ and define

$$\langle \langle \sigma, \eta \rangle \rangle_s = \int_X \langle (1 + \Delta)^s \sigma(x), \eta(x) \rangle dx,$$

where Δ is the Laplacian operator. Then the Soblev space $\mathcal{H}^s(X, \mathcal{V})$ is the completion of $\Gamma^\infty(X, \mathcal{V})$ under the norm $\langle \langle \cdot, \cdot \rangle \rangle_s$. Notice that $\mathcal{H}^s(X, \mathcal{V})$ is a Hilbert module over \mathcal{A} . Equivalently, $\langle \langle \cdot, \cdot \rangle \rangle_s$ can also be defined through Fourier transform as in the classical case. In the following, we adopt the convention that the measure on \mathbb{R}^n is the Lebesgue measure with an additional normalizing factor $(2\pi)^{-n/2}$.

Lemma A.1. *Let T be a pseudodifferential operator of order n*

$$T : \Gamma^\infty(X, \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{V}).$$

Then

$$\langle \langle T\sigma, T\sigma \rangle \rangle_{s-n} \leq C \langle \langle \sigma, \sigma \rangle \rangle_s$$

for all $\sigma \in \mathcal{H}^s(X, \mathcal{V})$.

Proof. The Fourier transform of $T\sigma$ is given by

$$\widehat{T\sigma}(\zeta) = \int e^{i\langle x, \xi - \zeta \rangle} t(x, \xi) \hat{\sigma}(\xi) d\xi dx$$

where $t(x, \xi)$ is the symbol of T . Define

$$q(\zeta, \xi) = \int e^{-i\langle x, \zeta \rangle} t(x, \xi) dx.$$

Then

$$\widehat{T\sigma}(\zeta) = \int q(\zeta - \xi, \xi) \hat{\sigma}(\xi) d\xi.$$

Define

$$K(\zeta, \xi) = q(\zeta - \xi, \xi) (1 + |\xi|)^{-s} (1 + |\zeta|)^{s-n}.$$

We now prove that

$$\langle\langle T\sigma, \eta \rangle\rangle_{s-n} + \langle\langle \eta, T\sigma \rangle\rangle_{s-n} \leq C_0 \left(\lambda \langle\langle \sigma, \sigma \rangle\rangle_s + \frac{1}{\lambda} \langle\langle \eta, \eta \rangle\rangle_{s-n} \right)$$

for all $\eta \in \mathcal{H}^{s-n}(X, \mathcal{V})$ and for all $\lambda > 0$. Here C_0 is some fixed constant independent of σ and η .

Without loss of generality, we reduce the proof to the case where \mathcal{V} is the trivial \mathcal{A} -bundle $X \times \mathcal{A}$ and $K(\zeta, \xi)$ is positive self-adjoint for all ξ and ζ . We have $K(\xi, \zeta) = (K^{1/2}(\xi, \zeta))^2$.

Notice that for $a, b \in \mathcal{A}$, we have

$$a^*b + ab^* \leq \lambda a^*a + \frac{1}{\lambda} b^*b$$

for all $\lambda > 0$. It follows that

$$\begin{aligned} & \langle\langle T\sigma, \eta \rangle\rangle_{s-n} + \langle\langle \eta, T\sigma \rangle\rangle_{s-n} \\ &= \int \hat{\sigma}^*(\xi) q^*(\zeta - \xi, \xi) \hat{\eta}(\zeta) (1 + |\zeta|)^{2(s-n)} d\xi d\zeta \\ & \quad + \int \hat{\eta}^*(\zeta) q(\zeta - \xi, \xi) \hat{\sigma}(\xi) (1 + |\zeta|)^{2(s-n)} d\xi d\zeta \\ &= \int \hat{\sigma}^*(\xi) K^{1/2}(\zeta, \xi) (1 + |\xi|)^s K^{1/2}(\zeta, \xi) \hat{\eta}(\zeta) (1 + |\zeta|)^{s-n} d\xi d\zeta \\ & \quad + \int \hat{\eta}^*(\zeta) K^{1/2}(\zeta, \xi) (1 + |\zeta|)^{s-n} K^{1/2}(\zeta, \xi) \hat{\sigma}(\xi) (1 + |\xi|)^s d\xi d\zeta \\ &\leq \lambda \int \hat{\sigma}^*(\xi) K(\zeta, \xi) \hat{\sigma}(\xi) (1 + |\xi|)^{2s} d\xi d\zeta \\ & \quad + \frac{1}{\lambda} \int \hat{\eta}^*(\zeta) K(\zeta, \xi) \hat{\eta}(\zeta) (1 + |\zeta|)^{2(s-n)} d\xi d\zeta \\ &\leq \lambda \int \|K(\zeta, \xi)\| \hat{\sigma}^*(\xi) \hat{\sigma}(\xi) (1 + |\xi|)^{2s} d\xi d\zeta \\ & \quad + \frac{1}{\lambda} \int \|K(\zeta, \xi)\| \hat{\eta}^*(\zeta) \hat{\eta}(\zeta) (1 + |\zeta|)^{2(s-n)} d\xi d\zeta. \end{aligned}$$

By standard estimates from (classical) pseudodifferential calculus, we have

$$\int \|K(\zeta, \xi)\| d\xi \leq C_0 \quad \text{and} \quad \int \|K(\zeta, \xi)\| d\zeta \leq C_0$$

for some constant C_0 . It follows that

$$\langle\langle T\sigma, \eta \rangle\rangle_{s-n} + \langle\langle \eta, T\sigma \rangle\rangle_{s-n} \leq C_0 \left(\lambda \langle\langle \sigma, \sigma \rangle\rangle_s + \frac{1}{\lambda} \langle\langle \eta, \eta \rangle\rangle_{s-n} \right)$$

for all $\eta \in \mathcal{H}^{s-n}(X, \mathcal{V})$ and for all $\lambda > 0$. The proof is finished by choosing $\lambda = C_0$ and $\eta = T\sigma$. \square

Lemma A.2. *Let P be an elliptic pseudodifferential operator of order d*

$$P : \Gamma^\infty(X, \mathcal{V}) \rightarrow \Gamma^\infty(X, \mathcal{V}).$$

Then

$$\langle\langle\sigma, \sigma\rangle\rangle_d \leq C\langle\langle\sigma, \sigma\rangle\rangle_0 + C\langle\langle P\sigma, P\sigma\rangle\rangle_0$$

for all $\sigma \in \mathcal{H}^d(X, \mathcal{V})$.

Proof. Let Q be a parametrix of P , that is, $1 - QP$ and $1 - PQ$ are smoothing operators. It follows from the previous lemma that

$$\begin{aligned} \langle\langle\sigma, \sigma\rangle\rangle_d &= \langle\langle(1 - QP)\sigma + QP\sigma, (1 - QP)\sigma + QP\sigma\rangle\rangle_d \\ &\leq 2\langle\langle(1 - QP)\sigma, (1 - QP)\sigma\rangle\rangle_d + 2\langle\langle QP\sigma, QP\sigma\rangle\rangle_d \\ &\leq C_1\langle\langle\sigma, \sigma\rangle\rangle_0 + C_2\langle\langle P\sigma, P\sigma\rangle\rangle_0 \end{aligned}$$

since $1 - QP$ is a smoothing operator and Q has order $-d$. This finishes the proof. \square

Let \mathcal{V} be an \mathcal{A} -bundle over \mathbb{R}^n . Consider $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ with coordinates $y \in \mathbb{R}^{n-k}$, $z \in \mathbb{R}^k$ and dual coordinates ξ, ζ . We define the restriction map $R : \Gamma^\infty(\mathbb{R}^n, \mathcal{V}) \rightarrow \Gamma^\infty(\mathbb{R}^{n-k}, \mathcal{V})$ by

$$R\sigma(y) = \sigma(y, 0).$$

We have the following lemma which generalizes the corresponding classical result (cf. [10, Theorem 6.9]) to the case of \mathcal{A} -bundles.

Lemma A.3. *If $s > k/2$, then we have*

$$\langle\langle R\sigma, R\sigma\rangle\rangle_{s-k/2} \leq C\langle\langle\sigma, \sigma\rangle\rangle_s$$

for all $\sigma \in \mathcal{H}^s(\mathbb{R}^n, \mathcal{V})$.

Proof. Without loss of generality, we assume $\mathcal{V} = \mathbb{R}^n \times \mathcal{A}$. It suffices to show that

$$\langle\langle R\sigma, R\sigma\rangle\rangle_{s-k/2} \leq C\langle\langle\sigma, \sigma\rangle\rangle_s$$

for all Schwartz sections σ . Notice that

$$\int e^{i\langle\xi, y\rangle} \widehat{R\sigma}(\xi) d\xi = R\sigma(y) = \sigma(y, 0) = \iint e^{i\langle\xi, y\rangle} \hat{\sigma}(\xi, \zeta) d\xi d\zeta$$

for all $y \in \mathbb{R}^{n-k}$. So $\widehat{R\sigma}(\xi) = \int \hat{\sigma}(\xi, \zeta) d\zeta$.

It follows that

$$\begin{aligned}
& 2\widehat{R\sigma}(\xi)\widehat{R\sigma}^*(\xi) \\
&= \iint \hat{\sigma}(\xi, \zeta_1)\hat{\sigma}^*(\xi, \zeta_2)d\zeta_1d\zeta_2 + \iint \hat{\sigma}(\xi, \zeta_2)\hat{\sigma}^*(\xi, \zeta_1)d\zeta_1d\zeta_2 \\
&= \iint a(\xi, \zeta_1, \zeta_2)b^*(\xi, \zeta_1, \zeta_2) + b(\xi, \zeta_1, \zeta_2)a^*(\xi, \zeta_1, \zeta_2)d\zeta_1d\zeta_2 \\
&\leq \iint a(\xi, \zeta_1, \zeta_2)a^*(\xi, \zeta_1, \zeta_2) + b(\xi, \zeta_1, \zeta_2)b^*(\xi, \zeta_1, \zeta_2)d\zeta_1d\zeta_2 \\
&= \iint \sigma(\xi, \zeta_1)\sigma^*(\xi, \zeta_1)(1+|\xi|+|\zeta_1|)^{2s}(1+|\xi|+|\zeta_2|)^{-2s}d\zeta_1d\zeta_2 \\
&\quad + \iint \sigma(\xi, \zeta_2)\sigma^*(\xi, \zeta_2)(1+|\xi|+|\zeta_2|)^{2s}(1+|\xi|+|\zeta_1|)^{-2s}d\zeta_1d\zeta_2,
\end{aligned}$$

where we denote by

$$\begin{aligned}
a(\xi, \zeta_1, \zeta_2) &= \hat{\sigma}(\xi, \zeta_1)(1+|\xi|+|\zeta_1|)^s(1+|\xi|+|\zeta_2|)^{-s}, \\
b(\xi, \zeta_1, \zeta_2) &= \hat{\sigma}(\xi, \zeta_2)(1+|\xi|+|\zeta_2|)^s(1+|\xi|+|\zeta_1|)^{-s}.
\end{aligned}$$

Notice that

$$\int (1+|\xi|+|\zeta|)^{-2s}d\zeta = C(1+|\xi|)^{k-2s}$$

for some constant C . Therefore, we have

$$2\widehat{R\sigma}(\xi)\widehat{R\sigma}^*(\xi) \leq 2C(1+|\xi|)^{k-2s} \iint \sigma(\xi, \zeta)\sigma^*(\xi, \zeta)(1+|\xi|+|\zeta|)^{2s}d\zeta.$$

Equivalently,

$$\widehat{R\sigma}(\xi)\widehat{R\sigma}^*(\xi)(1+|\xi|)^{2s-k} \leq C \iint \sigma(\xi, \zeta)\sigma^*(\xi, \zeta)(1+|\xi|+|\zeta|)^{2s}d\zeta.$$

Now the lemma follows by integrating both sides with respect to ξ . \square

Corollary A.4. *Let Ω be a bounded domain in \mathbb{R}^n with C^∞ boundary $\partial\Omega$ and $\ell \geq 1$. Define the restriction map*

$$R(\sigma) = \sigma|_{\partial\Omega} : C^\ell(\overline{\Omega}, \mathcal{V}) \rightarrow C^\ell(\partial\Omega, \mathcal{V}).$$

Then

$$\langle\langle R\sigma, R\sigma \rangle\rangle_{\ell-1/2} \leq C\langle\langle \sigma, \sigma \rangle\rangle_\ell$$

for all $\sigma \in \mathcal{H}^\ell(\Omega, \mathcal{V})$.

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